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TRANSACTIONS

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VOLUME VII. PART I.

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M.DCCC.XXXIX.

I. *On the Laws of the Reflexion and Refraction of Light at the common Surface of two non-crystallized Media. By* GEORGE GREEN, ESQ., B.A., *Caius College.*

[Read *December* 11, 1837.]

M. CAUCHY seems to have been the first who saw fully the utility of applying to the Theory of Light those formulæ which represent the motions of a system of molecules acting on each other by mutually attractive and repulsive forces; supposing always that in the mutual action of any two particles, the particles may be regarded as points animated by forces directed along the right line which joins them. This last supposition, if applied to those compound particles, at least, which are separable by mechanical division, seems rather restrictive; as many phenomena, those of crystallization for instance, seem to indicate certain polarities in these particles. If, however, this were not the case, we are so perfectly ignorant of the mode of action of the elements of the luminiferous ether on each other, that it would seem a safer method to take some general physical principle as the basis of our reasoning, rather than assume certain modes of action, which, after all, may be widely different from the mechanism employed by nature; more especially if this principle include in itself, as a particular case, those before used by M. Cauchy and others, and also lead to a much more simple process of calculation. The principle selected as the basis of the reasoning contained in the following paper is this: In whatever way the elements of any material system may act upon each other, if all the internal forces exerted be multiplied by the elements of their respective directions, the total sum for any assigned portion of the mass will always be the exact differential of some function. But, this function being known, we can immediately apply the general method given in the *Mécanique Analytique*, and which appears to be more especially applicable to problems that relate to the motions of systems composed of an immense number of particles mutually acting upon each other. One of the advantages of this method, of great importance, is, that we are necessarily led by the mere process of the calculation, and with little care on our part, to all the equations and conditions which are *requisite* and *sufficient* for the complete solution of any problem to which it may be applied.

The present communication is confined almost entirely to the consideration of non-crystallized media; for which it is proved, that the function due to the molecular actions, in its most general form, contains only two arbitrary coefficients, A and B; the values of which depend of course on the unknown internal constitution of the medium under consideration, and it would be easy to shew, for the most general case, that any arbitrary disturbance, excited in a very small portion of the medium, would in general, give rise to two spherical waves, one propagated entirely by normal, the other entirely by transverse, vibrations, and such that if the velocity of transmission of the former wave be represented by $\sqrt{A}$, that of the latter would be represented by $\sqrt{B}$. But in the transmission of light through a prism, though the wave which is propagated by normal vibrations were incapable itself of affecting the eye, yet it would be capable of giving rise to an ordinary wave of light propagated by transverse vibrations, except in the extreme cases where $\frac{A}{B} = 0$, or $\frac{A}{B} =$ a very large quantity; which, for the sake of simplicity, may be regarded as infinite; and it is not difficult to prove, that the equilibrium of our medium would be unstable unless $\frac{A}{B} > \frac{4}{3}$. We are therefore compelled to adopt the latter value of $\frac{A}{B}$, and thus to admit that in the luminiferous ether, the velocity of transmission of waves propagated by normal vibrations, is very great compared with that of ordinary light.

The principal results obtained in this paper, relate to the intensity of the waves reflected at the common surface of two media, both for light polarized in and perpendicular to the plane of incidence; and likewise to the change of phase which takes place when the reflexion becomes total. In the former case, our values agree precisely with those given by Fresnel; supposing, as he has done, that the direction of the actual motion of the particles of the luminiferous ether, is perpendicular to the plane of polarization. But it results from our formulæ, when the light is polarized perpendicular to the plane of incidence, that the expressions given by Fresnel are only very near approximations; and that the intensity of the reflected wave will never become absolutely null, but only attain a minimum value; which, in the case of reflexion from water at the proper angle, is $\frac{1}{151}$ part of that of the incident wave. This minimum value increases rapidly, as the index of refraction increases, and thus the quantity of light reflected at the polarizing angle, becomes considerable for highly refracting substances, a fact which has been long known to experimental philosophers.

It may be proper to observe, that M. Cauchy (*Bulletin des Sciences*, 1830,) has given a method of determining the intensity of the waves reflected at the common surface of two media. He has since stated, (*Nouveaux Exercices des Mathématiques*,) that the hypothesis employed on that occasion is inadmissible, and has promised in a future memoir, to give a *new mechanical principle* applicable to this and other questions; but I have not been able to learn whether such a memoir has yet appeared. The first method consisted in satisfying a part, and only a part, of the conditions belonging to the surface of junction, and the consideration of the waves propagated by normal vibrations was wholly overlooked, though it is easy to perceive, that in general waves of this kind must necessarily be produced when the incident wave is polarized perpendicular to the plane of incidence, in consequence of the incident and refracted waves being in different planes. Indeed, without introducing the consideration of these last waves, it is impossible to satisfy the whole of the conditions due to the surface of junction of the two media. But when this consideration is introduced, the whole of the conditions may be satisfied, and the principles given in the *Mécanique Analytique* became abundantly sufficient for the solution of the problem.

In conclusion, it may be observed, that the radius of the sphere of sensible action of the molecular forces has been regarded as unsensible with respect to the length λ of a wave of light, and thus, for the sake of simplicity, certain terms have been disregarded on which the different refrangibility of differently coloured rays might be supposed to depend. These terms, which are necessary to be considered when we are treating of the dispersion, serve only to render our formulæ uselessly complex in other investigations respecting the phenomena of light.

Let us conceive a mass composed of an immense number of molecules acting on each other by any kind of molecular forces, but which are sensible only at insensible distances, and let moreover the whole system be quite free from all extraneous action of every kind. Then x y and z being the co-ordinates of any particle of the medium under consideration when in equilibrium, and

$$x + u, \qquad y + v, \qquad z + w,$$

the co-ordinates of the same particle in a state of motion (where u, v, and w are very small functions of the original co-ordinates (x, y, z) of any particle and of the time (t)), we get, by combining D'Alembert's principle with that of virtual velocities,

$$\Sigma Dm \left\{ \frac{d^2 u}{dt^2}\, \delta u + \frac{d^2 v}{dt^2}\, \delta v + \frac{d^2 w}{dt^2}\, \delta w \right\} = \Sigma Dv \,.\, \delta\phi \quad (1);$$

Dm and Dv being exceedingly small corresponding elements of the mass and volume of the medium, but which nevertheless contain a very great number of molecules, and $\delta\phi$ the exact differential of some function and entirely due to the internal actions of the particles of the medium on each other. Indeed, if $\delta\phi$ were not an exact differential, a perpetual motion would be possible, and we have every reason to think, that the forces in nature are so disposed as to render this a natural impossibility.

Let us now take any element of the medium, rectangular in a state of repose, and of which the sides are dx, dy, dz the length of the sides composed of the same particles will in a state of motion become

$$dx' = dx\,(1+s_1), \quad dy' = dy\,(1+s_2), \quad dz' = dz\,(1+s_3);$$

where s_1, s_2, s_3 are exceedingly small quantities of the first order. If, moreover, we make

$$\alpha = \cos < \frac{dy'}{dz'}, \quad \beta = \cos < \frac{dx'}{dz'}, \quad \gamma = \cos < \frac{dx'}{dy'};$$

α, β and γ will be very small quantities of the same order. But, whatever may be the nature of the internal actions, if we represent by

$$\delta\phi\,dx\,dy\,dz,$$

the part of the second member of the equation (1), due to the molecules in the element under consideration, it is evident, that ϕ will remain the same when all the sides and all the angles of the parallelopiped, whose sides are $dx'\,dy'\,dz'$, remain unaltered, and therefore its most general value must be of the form

$$\phi = \text{function } \{s_1,\ s_2,\ s_3,\ \alpha,\ \beta,\ \gamma\}.$$

But s_1, s_2, s_3, α, β, γ being very small quantities of the first order, we may expand ϕ in a very convergent series of the form

$$\phi = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \text{\&c.}:$$

ϕ_0, ϕ_1, ϕ_2, &c. being homogeneous functions of the six quantities α, β, γ, s_1, s_2, s_3 of the degrees 0, 1, 2, &c. each of which is very great compared with the next following one. If now, ρ represent the primitive density of the element $dx\ dy\ dz$, we may write $\rho\ dx\ dy\ dz$ in the place of Dm in the formula (1), which will thus become, since ϕ_0 is constant,

$$\iiint \rho\,dx\,dy\,dz \left\{\frac{d^2u}{dt^2}\,\delta u + \frac{d^2v}{dt^2}\,\delta v + \frac{d^2w}{dt^2}\,\delta w\right\}$$
$$= \iiint dx\,dy\,dz\,(\delta\phi_1 + \delta\phi_2 + \text{\&c.});$$

the triple integrals extending over the whole volume of the medium under consideration.

But by the supposition, when $u=0$, $v=0$ and $w=0$, the system is in equilibrium, and hence

$$0 = \iiint dx\,dy\,dz\,\delta\phi_1:$$

seeing that ϕ_1 is a homogeneous function of s_1, s_2, s_3, α, β, γ of the *first* degree only. If therefore we neglect ϕ_3, ϕ_4, &c. which are exceedingly small compared with ϕ_2, our equation becomes

$$\iiint \rho\,dx\,dy\,dz \left\{\frac{d^2u}{dt^2}\delta u + \frac{d^2v}{dt^2}\delta v + \frac{d^2w}{dt^2}\delta w\right\} = \iiint dx\,dy\,dz\,\delta\phi_2 \quad (2);$$

the integrals extending over the whole volume under consideration. The formula just found is true for any number of media comprised in this volume, provided the whole system be perfectly free from all extraneous forces, and subject only to its own molecular actions.

If now we can obtain the value of ϕ_2, we shall only have to apply the general methods given in the *Mécanique Analytique*. But ϕ_2 being a homogeneous function of six quantities of the second degree, will in its most general form contain 21 arbitrary coefficients. The proper value to be assigned to each, will of course depend on the internal constitution of the medium. If, however, the medium be a non-crystallized one, the form of ϕ_2 will remain the same, whatever be the directions of the co-ordinate axes in space. Applying this last consideration, we shall find that the most general form of ϕ_2 for non-crystallized bodies contains only two arbitrary coefficients. In fact, by neglecting quantities of the higher orders, it is easy to perceive that

$$s_1 = \frac{du}{dx}, \quad s_2 = \frac{dv}{dy}, \quad s_3 = \frac{dw}{dz},$$

$$\alpha = \frac{dw}{dy} + \frac{dv}{dz}, \quad \beta = \frac{dw}{dx} + \frac{du}{dz}, \quad \gamma = \frac{du}{dy} + \frac{dv}{dx};$$

and if the medium is symmetrical with regard to the plane (xy) only, ϕ_2 will remain unchanged when $-z$ and $-w$ are written for z and w. But this alteration evidently changes α and β to $-\alpha$ and $-\beta$. Similar observations apply to the planes (xz) (yz), If therefore the medium is merely symmetrical with respect to each of the three co-ordinate planes, we see that ϕ_2 must remain unaltered when

$$\left.\begin{matrix} \text{or} & -z, & -w, & -\alpha, & -\beta \\ \text{or} & -y, & -v, & -\alpha, & -\gamma \\ \text{or} & -x, & -u, & -\beta, & -\gamma \end{matrix}\right\} \text{ are written for } \left\{\begin{matrix} z, & w, & \alpha, & \beta \\ y, & v, & \alpha, & \gamma \\ x, & u, & \beta, & \gamma. \end{matrix}\right.$$

In this way the 21 coefficients are reduced to 9, and the resulting function is of the form

$$G\left(\frac{du}{dx}\right)^2 + H\left(\frac{dv}{dy}\right)^2 + I\left(\frac{dw}{dz}\right)^2 + L\alpha^2 + M\beta^2 + N\gamma^2$$

$$+ 2P\frac{dv}{dy}\cdot\frac{dw}{dz} + 2Q\frac{du}{dx}\cdot\frac{dw}{dz} + 2R\frac{du}{dx}\cdot\frac{dv}{dy} = \phi_2. \qquad (A).$$

Probably the function just obtained may belong to those crystals which have three axes of elasticity at right angles to each other.

Suppose now we further restrict the generality of our function by making it symmetrical all round one axis, as that of z for instance. By shifting the axis of x through the infinitely small angle $\delta\theta$;

$$\left.\begin{matrix} x \\ y \\ z \end{matrix}\right\} \text{ becomes } \left\{\begin{matrix} x + y\,\delta\theta \\ y - x\,\delta\theta, \\ z \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{matrix}\right\} \text{ becomes } \left\{\begin{matrix} \frac{d}{dx} + \delta\theta\frac{d}{dy} \\ \frac{d}{dy} - d\theta\frac{d}{dx}, \\ \frac{d}{dz} \end{matrix}\right.$$

and

$$\left.\begin{matrix} u \\ v \\ w \end{matrix}\right\} \text{ becomes } \left\{\begin{matrix} u + v\,\delta\theta \\ v - u\,\delta\theta. \\ w \end{matrix}\right.$$

Making these substitutions in (A), we see that the form of ϕ_2 will not remain the same for the new axes, unless

$$G = H = 2N + R,$$

$$L = M,$$

$$P = Q;$$

and thus we get

$$\phi_2 = G\left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2\right\} + I\left(\frac{dw}{dz}\right)^2 + L(\alpha^2+\beta^2)$$
$$+ N\gamma^2 + 2P\left(\frac{dv}{dy} + \frac{du}{dx}\right)\frac{dw}{dz} + (2G-4N)\frac{du}{dx}\cdot\frac{dv}{dy}; \qquad (B).$$

under which form it may possibly be applied to uniaxal crystals.

Lastly, if we suppose the function ϕ_2 symmetrical with respect to all three axes, there results

$$G = H = I = 2N + R,$$
$$L = M = N,$$
$$P = Q = R;$$

and consequently,

$$\phi_2 = G\left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dz}\right)^2\right\} + L(\alpha^2+\beta^2+\gamma^2)$$
$$+ (2G-4L)\left\{\frac{dv}{dy}\cdot\frac{dw}{dz} + \frac{du}{dx}\cdot\frac{dw}{dz} + \frac{du}{dx}\cdot\frac{dv}{dy}\right\};$$

or, by merely changing the two constants and restoring the values of α, β, and γ,

$$2\phi_2 = -A\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)^2 \qquad (C).$$
$$-B\left\{\left(\frac{du}{dy} + \frac{dv}{dx}\right)^2 + \left(\frac{du}{dz} + \frac{dw}{dx}\right)^2 + \left(\frac{dv}{dz} + \frac{dw}{dy}\right)^2 - 4\left(\frac{dv}{dy}\cdot\frac{dw}{dz} + \frac{du}{dx}\cdot\frac{dw}{dz} + \frac{du}{dx}\cdot\frac{dv}{dy}\right)\right\}.$$

This is the most general form that ϕ_2 can take for non-crystallized bodies, in which it is perfectly indifferent in what directions the rectangular axes are placed. The same result might be obtained from the most general value of ϕ_2, by the method before used to make ϕ_2 symmetrical all round the axes of z, applied also to the other two axes. It was, indeed, thus I first obtained it. The method given in the text, however, and which is very similar to one used by M. Cauchy, is not only more simple, but has the advantage of furnishing two intermediate results, which may possibly be of use on some future occasion.

Let us now consider the particular case of two indefinitely extended media, the surface of junction when in equilibrium being a plane of infinite extent, horizontal (suppose), and which we shall take as that of (yz), and conceive the axis of x positive directed downwards. Then if ρ be the constant density of the upper, and $\rho_{,}$ that of the lower medium, ϕ_2 and $\phi_2^{(1)}$ the corresponding functions due to the molecular actions. The equation (2) adapted to the present case will become

$$\iiint \rho\, dx\, dy\, dz \left\{\frac{d^2 u}{dt^2}\delta u + \frac{d^2 v}{dt^2}\delta v + \frac{d^2 w}{dt^2}\delta w\right\}$$

$$+ \iiint \rho_{,}\, dx\, dy\, dz \left\{\frac{d^2 u_{,}}{dt^2}\delta u_{,} + \frac{d^2 v_{,}}{dt^2}\delta v_{,} + \frac{d^2 w_{,}}{dt^2}\delta w_{,}\right\}, \quad (3).$$

$$= \iiint dx\, dy\, dz\, \phi_2 + \iiint dx\, dy\, dz\, \phi_2^{(1)};$$

$u_{,}$, $v_{,}$, $w_{,}$ belonging to the lower fluid, and the triple integrals being extended over the whole volume of the fluids to which they respectively belong.

It now only remains to substitute for ϕ_2 and $\phi_2^{(1)}$ their values, to effect the integrations by parts, and to equate separately to zero the coefficients of the independent variations. Substituting therefore for ϕ_2 its value (C), we get

$$\iiint dx\, dy\, dz\, \delta\phi_2$$

$$= - A \iiint dx\, dy\, dz \left\{\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\left(\frac{d\delta u}{dx} + \frac{d\delta v}{dy} + \frac{d\delta w}{dz}\right)\right\}$$

$$- B \iiint dx\, dy\, dz \left\{\left(\frac{du}{dy} + \frac{dv}{dx}\right)\left(\frac{d\delta u}{dy} + \frac{d\delta v}{dx}\right) + \left(\frac{du}{dz} + \frac{dw}{dx}\right)\left(\frac{d\delta u}{dz} + \frac{d\delta w}{dx}\right)\right.$$

$$+ \left(\frac{dv}{dz} + \frac{dw}{dy}\right)\left(\frac{d\delta v}{dz} + \frac{d\delta w}{dy}\right) - 2\left[\left(\frac{dv}{dy}\cdot\frac{d\delta w}{dz} + \frac{dw}{dz}\cdot\frac{d\delta v}{dy}\right)\right.$$

$$\left.\left. + \left(\frac{du}{dx}\cdot\frac{d\delta w}{dz} + \frac{dw}{dz}\cdot\frac{d\delta u}{dx}\right) + \left(\frac{du}{dx}\,\frac{d\delta v}{dy} + \frac{dv}{dy}\cdot\frac{d\delta u}{dx}\right)\right]\right\}$$

$$= - \iint dy\,dz \left\{ A \cdot \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) - 2B \left(\frac{dv}{dy} + \frac{dw}{dz}\right)\right\} . \delta u$$

$$- \iint dy\,dz \left\{ B \left(\frac{du}{dy} + \frac{dv}{dx}\right) \delta v + B \left(\frac{du}{dz} + \frac{dw}{dx}\right) \delta w \right\}$$

$$+ \iiint dx\,dy\,dz \left\{ A \frac{d}{dx} \cdot \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + B \left[\frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} - \frac{d}{dx}\left(\frac{dv}{dy} + \frac{dw}{dz}\right)\right]\right\} . \delta u$$

$$+ \left\{ A \frac{d}{dy} \cdot \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + B \left[\frac{d^2v}{dx^2} + \frac{d^2v}{dz^2} - \frac{d}{dy}\left(\frac{du}{dx} + \frac{dw}{dz}\right)\right]\right\} \delta v$$

$$+ \left\{ A \frac{d}{dz} \cdot \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + B \left[\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} - \frac{d}{dz} \cdot \left(\frac{du}{dx} + \frac{dv}{dy}\right)\right]\right\} \delta w ;$$

seeing that we may neglect the double integrals at the limits $x = -\infty$, $y = \pm\infty$, $z = \pm\infty$; as the conditions imposed at these limits cannot affect the motion of the system at any *finite* distance from the origin; and thus the double integrals belong only to the surface of junction, of which the equation, in a state of equilibrium, is

$$0 = x.$$

In like manner we get

$$\iiint dx\,dy\,dz\,\delta\phi_2^{(1)}$$

$$= + \iint dy\,dz \left\{ A_, \left(\frac{du_,}{dx} + \frac{dv_,}{dy} + \frac{dw_,}{dz}\right) - 2B_, \left(\frac{dv_,}{dy} + \frac{dw_,}{dz}\right)\right\} \delta u_,$$

$$+ \iint dy\,dz \left\{ B \left(\frac{du_,}{dy} + \frac{dv_,}{dx}\right) \delta v_, + B \left(\frac{du_,}{dz} + \frac{dw_,}{dx}\right) \delta w_, \right\}$$

+ the triple integral;

since it is the *least* value of x which belongs to the surface of junction in the *lower* medium, and therefore the double integrals belonging to the limiting surface, must have their signs changed.

If, now, we substitute the preceding expression in (3), equate separately to zero the coefficients of the independent variation δu, δv, δw, under the triple sign of integration, there results for the upper medium

$$\rho \frac{d^2 u}{dt^2} = A \frac{d}{dx} \cdot \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + B \left\{\frac{d^2 u}{dy^2} + \frac{d^2 u}{dz^2} - \frac{d}{dx} \cdot \left(\frac{dv}{dy} + \frac{dw}{dz}\right)\right\};$$

$$\rho \frac{d^2 v}{dt^2} = A \frac{d}{dy} \cdot \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + B \left\{\frac{d^2 v}{dx^2} + \frac{d^2 v}{dz^2} - \frac{d}{dy} \cdot \left(\frac{du}{dx} + \frac{dw}{dz}\right)\right\}, \quad (4);$$

$$\rho \frac{d^2 w}{dt^2} = A \frac{d}{dz} \cdot \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + B \left\{\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} - \frac{d}{dz} \cdot \left(\frac{du}{dx} + \frac{dv}{dy}\right)\right\};$$

and by equating the coefficients of $\delta u_{,}$, $\delta v_{,}$, $\delta w_{,}$, we get three similar equations for the lower medium.

To the six general equations just obtained, we must add the conditions due to the surface of junction of the two media; and at this surface we have first,

$$u = u_{,}, \quad v = v_{,}, \quad w = w_{,}, \quad (\text{when } x = 0), \quad (5);$$

and consequently,

$$\delta u = \delta u_{,}; \quad \delta v = \delta v_{,}; \quad \delta w = \delta w_{,}.$$

But the part of the equation (3) belonging to this surface, and which yet remains to be satisfied, is

$$\begin{aligned} 0 = &- \iint dy\, dz \left\{A \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) - 2B \left(\frac{dv}{dy} + \frac{dw}{dz}\right)\right\} \delta u \\ &+ \iint dy\, dz \left\{A_{,} \left(\frac{du_{,}}{dx} + \frac{dv_{,}}{dy} + \frac{dw_{,}}{dz}\right) - 2B_{,} \left(\frac{dv_{,}}{dy} + \frac{dw_{,}}{dz}\right)\right\} \delta u_{,} \\ &- \iint dy\, dz \left\{B \left(\frac{du}{dy} + \frac{dv}{dx}\right) \delta v + B \left(\frac{du}{dz} + \frac{dw}{dx}\right) \delta w\right\} \\ &+ \iint dy\, dz \left\{B_{,} \left(\frac{du_{,}}{dy} + \frac{dv_{,}}{dx}\right) \delta v_{,} + B_{,} \left(\frac{du_{,}}{dz} + \frac{dw_{,}}{dx}\right) \delta w_{,}\right\}; \end{aligned}$$

and as $\delta u = \delta u_{,}$, &c., we obtain, as before,

$$A \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) - 2B \left(\frac{dv}{dy} + \frac{dw}{dz}\right) = A_{,} \left(\frac{du_{,}}{dx} + \frac{dv_{,}}{dy} + \frac{dw_{,}}{dz}\right) - 2B_{,} \left(\frac{dv_{,}}{dy} + \frac{dw_{,}}{dz}\right)$$

$$B \left(\frac{du}{dy} + \frac{dv}{dx}\right) = B_{,} \left(\frac{du_{,}}{dy} + \frac{dv_{,}}{dx}\right), \quad (6)$$

$$B \left(\frac{du}{dz} + \frac{dw}{dx}\right) = B_{,} \left(\frac{du_{,}}{dz} + \frac{dw_{,}}{dx}\right);$$

and these belong to the particular value $x = 0$.

The six particular conditions (5) and (6), belonging to the surface of junction of the two media, combined with the six general equations before obtained, are *necessary* and *sufficient* for the complete determination of the motion of the two media, supposing the initial state of each given. We shall not here attempt their general solution, but merely consider the propagation of a plane wave of infinite extent, accompanied by its reflected and refracted waves, as in the preceding paper on Sound.

Let the direction of the axis of z, which yet remains arbitrary, be taken parallel to the intersection of the plane of the incident wave with the surface of junction, and suppose the disturbance of the particles to be wholly in the direction of the axis of z, which is the case with light polarized in the plane of incidence, according to Fresnel. Then we have

$$0 = u, \quad 0 = v, \quad 0 = u_{,}, \quad 0 = v_{,};$$

and supposing the disturbance the same for every point of the same front of a wave, w and $w_{,}$ will be independent of z, and thus the three general equations (4), will all be satisfied, if

$$\rho \frac{d^2 w}{dt^2} = B \left\{ \frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} \right\},$$

or by making $\dfrac{B}{\rho} = \gamma^2$,

$$\frac{d^2 w}{dt^2} = \gamma^2 \left\{ \frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} \right\}, \quad (7).$$

Similarly in the lower medium we have

$$\frac{d^2 w_{,}}{dt^2} = \gamma_{,}^2 \left\{ \frac{d^2 w_{,}}{dx^2} + \frac{d^2 w_{,}}{dy^2} \right\}, \quad (8),$$

$w_{,}$ and $\gamma_{,}$ belonging to this medium.

It now remains to satisfy the conditions (5) and (6). But these are all satisfied by the preceding values provided,

$$w = w_{,},$$

$$B \frac{dw}{dx} = B_{,} \frac{dw_{,}}{dx}.$$

The formulæ which we have obtained are quite general, and will apply to the ordinary elastic fluids by making $B = 0$. But for all the known gases, A is independent of the nature of the gas, and consequently $A = A_{,}$. If, therefore, we suppose $B = B_{,}$, at least when we consider those phenomena only which depend merely on different states of the same medium, as is the case with light, our conditions become*

$$\left.\begin{aligned} w &= w_{,} \\ \frac{dw}{dx} &= \frac{dw_{,}}{dx} \end{aligned}\right\} \text{(when } x=0\text{)}, \qquad (9).$$

The disturbance in the upper medium which contains the incident and reflected wave, will be represented, as in the case of Sound, by

$$w = f(ax + by + ct) + F(-ax + by + ct);$$

f belonging to the incident, F to the reflected plane wave, and c being a negative quantity. Also in the lower medium,

$$w_{,} = f_{,}(a_{,}x + by + ct).$$

These values evidently satisfy the general equation (7) and (8), provided $c^2 = \gamma^2 (a^2 + b^2)$, and $c^2 = \gamma_{,}^2 (a_{,}^2 + b^2)$; we have therefore only to satisfy the conditions (9), which give

$$f(by + ct) + F(by + ct) = f_{,}(by + ct),$$

$$af'(by + ct) - aF'(by + ct) = a_{,}f_{,}'(by + ct).$$

Taking now the differential coefficient of the first equation, and writing to abridge the characteristics of the functions only, we get

$$2f' = \left(1 + \frac{a_{,}}{a}\right) f_{,}', \text{ and } 2F' = \left(1 - \frac{a_{,}}{a}\right) f_{,}',$$

* Though for all known gases A is independent of the nature of the gas, perhaps it is extending the analogy rather too far, to assume that in the luminiferous ether the constants A and B must always be independent of the state of the ether, as found in different refracting substances. However, since this hypothesis greatly simplifies the equations due to the surface of junction of the two media, and is itself the most simple that could be selected, it seemed natural, first to deduce the consequences which follow from it before trying a more complicated one, and, as far as I have yet found, these consequences are in accordance with observed facts.

and therefore

$$\frac{F'}{f'} = \frac{1 - \frac{a_{,}}{a}}{1 + \frac{a_{,}}{a}} = \frac{a - a_{,}}{a + a_{,}} = \frac{\cot\theta - \cot\theta_{,}}{\cot\theta + \cot\theta_{,}} = \frac{\sin(\theta_{,} - \theta)}{\sin(\theta_{,} + \theta)};$$

θ and $\theta_{,}$ being the angles of incidence and refraction.

This ratio between the intensity of the incident and reflected waves, is exactly the same as that for light polarized in the plane of incidence, (vide Airy's *Tracts*, p. 356,) and which Fresnel supposes to be propagated by vibrations perpendicular to the plane of incidence, agreeably to what has been assumed in the foregoing process.

We will now limit the generality of the functions f, F and $f_{,}$, by supposing the law of the motion to be similar to that of a cycloidal pendulum; and if we farther suppose the angle of incidence to be increased until the refracted wave ceases to be transmitted in the regular way, as in our former paper on Sound, the proper integral of the equation

$$\frac{d^2 w_{,}}{dt^2} = \gamma_{,}^2 \left\{ \frac{d^2 w_{,}}{dx^2} + \frac{d^2 w_{,}}{dy^2} \right\},$$

will be

$$w_{,} = \epsilon^{-a_{,}'x} B \sin\psi, \quad (10);$$

where $\psi = by + ct$, and $a_{,}'$ is determined by

$$\gamma_{,}^2 (b^2 - a_{,}'^2) = c^2 = \gamma^2 (b^2 + a^2), \quad (11).$$

But one of the conditions (9) will introduce *sines* and the other *cosines*, in such a way that it will be impossible to satisfy them unless we introduce both *sines* and *cosines* into the value of w, or, which amounts to the same, unless we make

$$w = \alpha \sin(ax + by + ct + e) + \beta \sin(-ax + by + ct + e_{,}), \quad (12),$$

in the first medium, instead of

$$w = \alpha \sin(ax + by + ct) + \beta \sin(-ax + by + ct),$$

which would have been done had the refracted wave been transmitted in the usual way, and consequently no exponential been introduced into the value of $w_,$. We thus see the analytical reason for what is called the change of phase which takes place when the reflexion of light becomes total.

Substituting now (10) and (12), in the equations (9), and proceeding precisely as for sound, we get

$$0 = \alpha \cos e - \beta \cos e_,,$$

$$0 = \alpha \sin e + \beta \sin e_,,$$

$$\frac{a_,'}{a} B = \alpha \sin e - \beta \sin e_,,$$

$$B = \alpha \cos e + \beta \cos e_,.$$

Hence there results $\alpha = \beta$, and $e_, = - e$, and

$$\tan e = \frac{a_,'}{a} = \frac{a_,'}{b} \div \frac{a}{b} = \frac{a_,'}{b} \tan \theta.$$

But by (11),

$$\frac{a_,'}{b} = \sqrt{\left\{1 - \frac{\gamma^2}{\gamma_,^2} \cdot \left(1 + \frac{a^2}{b^2}\right)\right\}} = \sqrt{\left(1 - \frac{1}{\mu^2 \sin^2 \theta}\right)};$$

by introducing μ the index of refraction, and θ the angle of incidence. Thus,

$$\tan e = \frac{\sqrt{(\mu^2 \sin^2 \theta - 1)}}{\mu \cos \theta};$$

and as e represents half the alteration of phase in passing from the incident to the reflected wave, we see that here also our result agrees precisely with Fresnel's, for light polarized in the plane of incidence. (Vide Airy's *Tracts*, p. 362.)

Let us now conceive the direction of the transverse vibrations in the incident wave to be perpendicular to the direction in the case just considered; and therefore that the actual motions of the particles are all parallel to the intersection of the plane of incidence (xy) with the front of the wave. Then, as the planes of the incident and refracted waves do not coincide, it is easy to perceive that at the surface of junction there will, in this case, be a resolved part of the disturbance in the direction of the normal; and therefore, besides the incident wave, there will, in general, be an accompanying reflected and refracted wave, in which the vibrations are transverse, and another pair of accompanying reflected and refracted waves, in which the directions of the vibrations are normal to the fronts of the waves. In fact, unless the consideration of the two latter waves is also introduced, it is impossible to satisfy all the conditions at the surface of junction; and these are as essential to the complete solution of the problem, as the general equations of motion.

The direction of the disturbance being in plane (xy) $w=0$, and as the disturbance of every particle in the same front of a wave is the same, u and v are independent of z. Hence, the general equations (4) for the first medium become

$$\frac{d^2u}{dt^2} = g^2 \frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy}\right) + \gamma^2 \frac{d}{dy}\left(\frac{du}{dy} - \frac{dv}{dx}\right),$$

$$\frac{d^2v}{dt^2} = g^2 \frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy}\right) + \gamma^2 \frac{d}{dx}\left(\frac{dv}{dx} - \frac{du}{dy}\right),$$

where $g^2 = \frac{A}{\rho}$, and $\gamma^2 = \frac{B}{\rho}$.

These equations might be immediately employed in their present form; but they will take a rather more simple form, by making

$$u = \frac{d\phi}{dx} + \frac{d\psi}{dy},$$
$$v = \frac{d\phi}{dy} - \frac{d\psi}{dx}; \qquad (13).$$

ϕ and ψ being two functions of x, y and t, to be determined.

By substitution, we readily see that the two preceding equations are equivalent to the system,

$$\frac{d^2\phi}{dt^2} = g^2\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right),$$
$$\frac{d^2\psi}{dt^2} = \gamma^2\left(\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2}\right). \qquad (14).$$

In like manner, if in the second medium we make

$$u_{,} = \frac{d\phi_{,}}{dx} + \frac{d\psi_{,}}{dy},$$
$$v_{,} = \frac{d\phi_{,}}{dy} - \frac{d\psi_{,}}{dx}, \qquad (15).$$

we get to determine $\phi_{,}$ and $\psi_{,}$ the equations

$$\frac{d^2\phi_{,}}{dt^2} = g_{,}^2\left(\frac{d^2\phi_{,}}{dx^2} + \frac{d^2\phi_{,}}{dy^2}\right),$$
$$\frac{d^2\psi_{,}}{dt^2} = \gamma_{,}^2\left(\frac{d^2\psi_{,}}{dx^2} + \frac{d^2\psi_{,}}{dy^2}\right), \qquad (16).$$

and as we suppose the constants A and B the same for both media, we have

$$\frac{\gamma}{\gamma_{,}} = \frac{g}{g_{,}}.$$

For the complete determination of the motion in question, it will be necessary to satisfy all the conditions due to the surface of junction of the two media. But, since $w = 0$ and $w_{,} = 0$, also, since u, v, $u_{,}$, $v_{,}$ are independent of z, the equations (5) and (6) become

$$u = u_{,}, \qquad v = v_{,};$$

$$A\left(\frac{du}{dx} + \frac{dv}{dy}\right) - 2B\frac{dv}{dy} = A\left(\frac{du_{,}}{dx} + \frac{dv_{,}}{dy}\right) - 2B\frac{dv_{,}}{dy},$$

$$\frac{du}{dy} + \frac{dv}{dx} = \frac{du_{,}}{dy} + \frac{dv_{,}}{dx},$$

provided $x = 0$. But since $x = 0$ in the last equations, we may differentiate them with regard to any of the independent variables except x, and thus the two latter, in consequence of the two former, will become

$$\frac{du}{dx} = \frac{du_{,}}{dx}, \qquad \frac{dv}{dx} = \frac{dv_{,}}{dx}.$$

Substituting now for u, v, &c., their values (13) and (15), in ϕ and ψ, the four resulting conditions relative to the surface of junction of the two media may be written,

$$\left.\begin{aligned}\frac{d\phi}{dx}+\frac{d\psi}{dy}&=\frac{d\phi_{,}}{dx}+\frac{d\psi_{,}}{dy}\\ \frac{d\phi}{dy}-\frac{d\psi}{dx}&=\frac{d\phi_{,}}{dy}-\frac{d\psi_{,}}{dx}\\ \frac{d^2\phi}{dx^2}+\frac{d^2\psi}{dx\,dy}&=\frac{d^2\phi_{,}}{dx^2}+\frac{d^2\psi_{,}}{dx\,dy}\\ \frac{d^2\phi}{dx\,dy}-\frac{d^2\psi}{dx^2}&=\frac{d^2\phi_{,}}{dx\,dy}-\frac{d^2\psi_{,}}{dx^2}\end{aligned}\right\}\text{(when } x=0\text{);}$$

or since we may differentiate with respect to y, the first and fourth equations give

$$\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}=\frac{d^2\psi_{,}}{dx^2}+\frac{d^2\psi_{,}}{dy^2};$$

in like manner, the second and third give

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=\frac{d^2\phi_{,}}{dx^2}+\frac{d^2\phi_{,}}{dy^2},$$

which, in consequence of the general equations (14) and (16), become

$$\frac{d^2\psi}{\gamma^2dt^2}=\frac{d^2\psi_{,}}{\gamma_{,}^2dt^2},\quad\text{and}\quad\frac{d^2\phi}{g^2dt^2}=\frac{d^2\phi_{,}}{g_{,}^2dt^2}.$$

Hence, the equivalent of the four conditions relative to the surface of junction, may be written

$$\left.\begin{aligned}\frac{d\phi}{dx}+\frac{d\psi}{dy}&=\frac{d\phi_{,}}{dx}+\frac{d\psi_{,}}{dy}\\ \frac{d\phi}{dy}-\frac{d\psi}{dx}&=\frac{d\phi_{,}}{dy}-\frac{d\psi_{,}}{dx}\\ \frac{d^2\phi}{g^2dt^2}&=\frac{d^2\phi_{,}}{g_{,}^2dt^2}\\ \frac{d^2\psi}{\gamma^2dt^2}&=\frac{d^2\psi_{,}}{\gamma_{,}^2dt^2}\end{aligned}\right\}\text{(when } x=0\text{),}\qquad (17).$$

If we examine the expressions (13) and (15), we shall see that the disturbances due to ϕ and $\phi_{,}$ are normal to the front of the wave to which they belong, whilst those which are due to ψ and $\psi_{,}$ are transverse or wholly in the front of the wave. If the coefficients A and B

did not differ greatly in magnitude, waves propagated by both kinds of vibrations must in general exist, as was before observed. In this case, we should have in the upper medium

$$\psi = f(ax + by + ct) + F(-ax + by + ct),$$

and (18).

$$\phi = \chi(-a'x + by + ct);$$

and for the lower one

$$\psi_{,} = f_{,}(a_{,}x + by + ct),$$

(19).

$$\phi_{,} = \chi_{,}(a_{,}'x + by + ct).$$

The coefficients b and c being the same for all the functions to simplify the results, since the indeterminate coefficients $a', a_{,}, a_{,}'$ will allow the fronts of the waves to which they respectively belong, to take any position that the nature of the problem may require. The coefficient of x in F belonging to that reflected wave, which, like the incident one, is propagated by transverse vibrations would have been determined exactly like $a', a_{,} a'$, as, however, it evidently $= -a$, it was for the sake of simplicity introduced immediately into our formulæ.

By substituting the values just given in the general equations (14) and (16), there results

$$c^2 = (a^2 + b^2)\gamma^2 = (a_{,}^2 + b^2)\gamma_{,}^2 = (a'^2 + b^2)g^2 = (a_{,}'^2 + b^2)g_{,}^2,$$

we have thus the position of the fronts of the reflected and refracted waves.

It now remains to satisfy the conditions due to the surface of junction of the two media. Substituting, therefore, the values (18) and (19) in the equations (17), we get

$$f'' + F'' = \frac{\gamma^2}{\gamma_{,}^2} f_{,}'',$$

$$\chi'' = \frac{g^2}{g_{,}^2}\chi_{,}'';$$

$$-a'\chi' + b(f' + F') = a_{,}'\chi_{,}' + bf_{,}',$$

$$b\chi' - a(f' - F') = b\chi_{,}' - a_{,}f_{,}';$$

where to abridge, the characteristics only of the functions are written.

By means of the last four equations, we shall readily get the values of $F''\chi''$ $f_{,}''\chi_{,}''$ in terms of f'', and thus obtain the intensities of the two reflected and two refracted waves, when the coefficients A and B do not differ greatly in magnitude, and the angle which the incident wave makes with the plane surface of junction is contained within certain limits. But in the introductory remarks, it was shewn that $\frac{A}{B}$ = a very great quantity, which may be regarded as infinite, and therefore g and $g_{,}$ may be regarded as infinite compared with γ and $\gamma_{,}$. Hence, for all angles of incidence except such as are infinitely small, the waves dependent on ϕ and $\phi_{,}$ cease to be transmitted in the regular way. We shall therefore, as before, restrain the generality of our functions, by supposing the law of the motion to be similar to that of a cycloidal pendulum, and as two of the waves cease to be transmitted in the regular way, we must suppose in the upper medium

$$\psi = \alpha \sin(ax + by + ct + e) + \beta \sin(-ax + by + ct + e_{,}),$$

and (20).

$$\phi = \epsilon^{a'x}(A \sin\psi_0 + B\cos\psi_0);$$

and in the lower one

$$\psi_{,} = \alpha_{,} \sin(a_{,}x + by + ct),$$

$$\phi_{,} = \epsilon^{-a_{,}'x}(A_{,}\sin\psi_0 + B_{,}\cos\psi_0), \quad (21).$$

where to abridge $\psi_0 = by + ct$.

These substituted in the general equations (14) and (15), give

$$c^2 = \gamma^2(a^2 + b^2) = \gamma_{,}^2(a_{,}^2 + b^2) = g^2(-a'^2 + b^2) = g_{,}^2(-a_{,}'^2 + b^2),$$

or, since g and $g_{,}$ are both infinite,

$$b = a' = a_{,}'.$$

It only remains to substitute the values (20) (21) in the equations (17), which belong to the surface of junction, and thus we get

$$bA\sin\psi_0 + bB\cos\psi_0 + b\alpha\cos(\psi_0 + e) + b\beta\cos(\psi_0 + e_{,})$$

$$= -bA_{,}\sin\psi_0 - bB_{,}\cos\psi_0 + b\alpha_{,}\cos\psi_0,$$

$$bA\cos\psi_0 - bB\sin\psi_0 - a\alpha\cos(\psi_0+e) + a\beta\cos(\psi_0+e_,)$$
$$= bA_,\cos\psi_0 - bB_,\sin\psi_0 - a_,\alpha_,\cos\psi_0, \qquad (22).$$
$$\frac{1}{g^2}(A\sin\psi_0 + B\cos\psi_0) = \frac{1}{g_,^2}(A_,\sin\psi_0 + B_,\cos\psi_0),$$
$$\frac{1}{\gamma^2}\{\alpha\sin(\psi_0+e) + \beta\sin(\psi_0+e_,)\} = \frac{1}{\gamma_,^2}\alpha_,\sin\psi_0.$$

Expanding the two last equations, comparing separately the coefficients of $\cos\psi_0$ and $\sin\psi_0$, and observing that

$$\frac{g}{g_,} = \frac{\gamma}{\gamma_,} = \mu \text{ suppose,}$$

we get

$$\left.\begin{aligned} A &= \mu^2 A_,, \\ B &= \mu^2 B_,, \\ \alpha\cos e + \beta\cos e_, &= \mu^2\alpha_,, \\ \alpha\sin e + \beta\sin e_, &= 0. \end{aligned}\right\} \qquad (23).$$

In like manner the two first equations of (22) will give

$$0 = A + A_, - \alpha\sin e - \beta\sin e_,,$$
$$0 = A - A_, + \frac{a_,\alpha_,}{b} + \frac{a}{b}(\beta\cos e_, - \alpha\cos e),$$
$$0 = B + B_, + \alpha\cos e + \beta\cos e_, - \alpha_,,$$
$$0 = B - B_, + \frac{a}{b}(\beta\sin e_, - \alpha\sin e);$$

combining these with the system (23), there results

$$\left.\begin{aligned} 0 &= A + A_,, \\ 0 &= B + B_, + (\mu^2 - 1)\alpha_,, \\ 0 &= A - A_, + \frac{a_,\alpha_,}{b} + \frac{a}{b}(\beta\cos e_, - \alpha\cos e), \\ 0 &= B - B_, + \frac{a}{b}(\beta\sin e_, - \alpha\sin e). \end{aligned}\right\} \qquad (24).$$

Again, the systems (23) and (24) readily give

$$\left.\begin{aligned}
\alpha \sin e &= -\tfrac{1}{2}\cdot\frac{(\mu^2-1)^2}{\mu^2+1}\,\frac{b}{a}\,\alpha_{,},\\
\alpha \cos e &= \tfrac{1}{2}\cdot\left(\mu^2+\frac{a_{,}}{a}\right)\alpha_{,},\\
\beta \sin e_{,} &= \tfrac{1}{2}\cdot\frac{(\mu^2-1)^2}{\mu^2+1}\,\frac{b}{a}\,\alpha_{,},\\
\beta \cos e_{,} &= \tfrac{1}{2}\cdot\left(\mu^2-\frac{a_{,}}{a}\right)\alpha_{,};
\end{aligned}\right\} \quad (25).$$

and therefore

$$\frac{\beta^2}{\alpha^2} = \frac{(\mu^2+1)^2\cdot\left(\mu^2-\dfrac{a_{,}}{a}\right)^2 + (\mu^2-1)^4\,\dfrac{b^2}{a^2}}{(\mu^2+1)^2\cdot\left(\mu^2+\dfrac{a_{,}}{a}\right)^2 + (\mu^2-1)^4\,\dfrac{b^2}{a^2}}, \quad (26).$$

When the refractive power in passing from the upper to the lower medium is not very great, μ does not differ much from 1. Hence, sin e and sin $e_{,}$ are small, and cos e, cos $e_{,}$ do not differ sensibly from unity; we have, therefore, as a first approximation,

$$\frac{\beta}{\alpha} = \frac{\mu^2-\dfrac{a_{,}}{a}}{\mu^2+\dfrac{a_{,}}{a}} = \frac{\dfrac{\sin^2\theta}{\sin^2\theta_{,}}-\dfrac{\cot\theta_{,}}{\cot\theta}}{\dfrac{\sin^2\theta}{\sin^2\theta_{,}}+\dfrac{\cot\theta_{,}}{\cot\theta}} = \frac{\sin 2\theta-\sin 2\theta_{,}}{\sin 2\theta+\sin 2\theta_{,}} = \frac{\tan(\theta-\theta_{,})}{\tan(\theta+\theta_{,})},$$

which agrees with the formula in Airy's *Tracts*, p. 358, for light polarized perpendicular to the plane of reflexion. This result is only a near approximation: but the formula (26) gives the correct value of $\dfrac{\beta^2}{\alpha^2}$, or the ratio of the intensity of the reflected to the incident light; supposing, with all optical writers, that the intensity of light is properly measured by the square of the actual velocity of the molecules of the luminiferous ether.

From the rigorous value (26), we see that the intensity of the reflected light never becomes absolutely null, but attains a minimum value nearly when

$$0 = \mu^2 - \frac{a_{\prime}}{a}, \text{ i.e., when } \tan(\theta + \theta_{\prime}) = \infty,$$

which agrees with experiment, and this minimum value is, since (27) gives $\frac{b}{a} = \mu$,

$$\frac{\beta^2}{\alpha^2} = \frac{(\mu^2 - 1)^4 \frac{b^2}{a^2}}{4(\mu^2 + 1)^2 \mu^4 + (\mu^2 - 1)^4 \frac{b^2}{a^2}} = \frac{(\mu^2 - 1)^4}{4\mu^2 (\mu^2 + 1)^2 + (\mu^2 - 1)^4}, \qquad (28).$$

If $\mu = \frac{4}{3}$, as when the two media are air and water, we get

$$\frac{\beta^2}{\alpha^2} = \frac{1}{151} \text{ nearly.}$$

It is evident from the formula (28), that the magnitude of this minimum value increases very rapidly as the index of refraction increases, so that for highly refracting substances, the intensity of the light reflected at the polarizing angle becomes very sensible, agreeably to what has been long since observed by experimental philosophers. Moreover, an inspection of the equations (25) will shew, that when we gradually increase the angle of incidence so as to pass through the polarizing angle, the change which takes place in the reflected wave is not due to an alteration of the sign of the coefficient β, but to a change of phase in the wave, which for ordinary refracting substances is very nearly equal to 180^0; the minimum value of β being so small as to cause the reflected wave sensibly to disappear. But in strongly refracting substances like diamond, the coefficient β remains so large that the reflected wave does not seem to vanish, and the change of phase is considerably less than 180^0. These results of our theory appear to agree with the observations of Professor Airy. (*Camb. Phil. Trans.* Vol. IV. p. 418., &c.)

Lastly, if the velocity $\gamma_{\prime}$ of transmission of a wave in the lower exceed γ that in the upper medium, we may, by sufficiently augmenting the angle of incidence, cause the refracted wave to disappear, and the change of phase thus produced in the reflected wave may readily be found. As the calculation is extremely easy after what precedes, it seems sufficient to give the result. Let therefore, here, $\mu = \frac{\gamma_{,}}{\gamma}$, also e, $e_{,}$ and θ as before, then $e_{,} = -e$, and the accurate value of e is given by

$$\tan e = \mu \sqrt{\mu^2 \tan^2 \theta - \sec^2 \theta} - \frac{(\mu^2 - 1)^2 \tan \theta}{\mu^2 + 1}.$$

The first term of this expression agrees with the formula of page 362 Airy's *Tracts*, and the second will be scarcely sensible except for highly refracting substances.

II. *On Molecular Equilibrium.* Part I. *By the Rev.* PHILIP KELLAND, M.A., *Queens' College, Cambridge; Professor of Mathematics in the University of Edinburgh.*

[Read *March* 26, 1838.]

INTRODUCTION.

1. WHATEVER ideas may have been entertained of the nature of forces at a distance from the centre of action, there appear to have been no very definite notions current respecting molecular forces, till within a few years from the present time. The obvious change in the attractions of the different parts of a solid body, produced by separating the particles by ever so small an interval; the fact that the attraction of cohesion when destroyed cannot be restored by any ordinary pressure, indicated that the force which the particles exert on each other in their positions of equilibrium, is of a nature totally distinct from the appreciable attractions and repulsions at finite distances. Newton only threw out hints respecting the nature of forces of this kind, never applying them, except in a popular manner in his Optics. One kind of molecular force which he conjectures is that of the particles of air and the magnetic ones, Newton applies to calculation, but he by no means supposes his hypothesis the correct one; on the contrary he appears to entertain great doubts on the subject, for he concludes his scholium by observing: "Whether elastic fluids do really consist of particles so repelling each other, is a physical question. We have demonstrated the properties of fluids consisting of particles of this kind, that hence philosophers may take occasion to discuss that question."

The phenomena of electricity and magnetism did indeed suggest hypotheses respecting the internal constitution of bodies, but these hypotheses, for the most part, were only partial ones. Those of Æpinus, Cavendish, and Franklin, fully establish a disposition of different sets of particles, but leave the *possibility* of such a disposition as consistent with the conditions of equilibrium to other hypotheses of a nature totally different from the one applied. With one or two exceptions it would appear, that all writers have regarded the molecular force as of a nature either distinct from that of the attractions and repulsions of the electric particles, or as the fundamental expression of which the law, in the latter case, is only a limiting form.

About the middle of the last century, however, Dr Knight published his "Attempt to explain all the Phenomena of Nature by means of two Principles, Attraction and Repulsion." The hypothesis adopted in this work, appears to be nearly the same as that usually adopted by theorists in Chemistry of the present day, and which is not essentially different from that which forms the basis of the present Memoir, with the exception, that the Author supposes the law of force to be the inverse power of the distance. Bodies are imagined to be formed of combinations of two groups of particles acting differently on each other, the one set mutually attractive, the other mutually repulsive; the former, by peculiar arrangements aggregated together, determine the nature of different substances; whilst the latter are collected around these groups, and form their atmospheres. I regret that I have not been able to meet with Dr Knight's work, which appears from the notices of it, to have been a sound and admirable treatise.

A few years later appeared Boscovich's "*Theoria Philosophiæ Naturalis ad unicam legem virium, in Naturâ existentium redacta,*" a work which from its title professes the reduction of all forces to one and the same law. As that law will be found to be the *conclusion* from another more simple law, I shall briefly state its principal features.

(1). "The atoms of matter are endued with attractive or repulsive forces to one another, of which the law of variation is the same for all."

(2). "Action and reaction are equal."

(3). "The nature of the force is such that at different distances it is attractive and repulsive alternately, so that a particle in receding from another, is first repelled, then attracted, then again repelled, and so on."

(4). "When the distance is indefinitely diminished, the force is repulsive and is indefinitely increased; and when the distance is indefinitely increased, the force is attractive and diminishes as the inverse square of the distance."

Such are the general features of Boscovich's law of molecular action. It will be our endeavour to deduce from an hypothesis not very different from that of Knight, a law resembling the above in its general features.

2. Notwithstanding the long interval that has elapsed since the publication of Boscovich's work, very little has been done on the subject, except by way of application, until very lately. Capillary attraction is a phenomenon, the solution of which, clearly requires a molecular hypothesis; but, unfortunately, the nature of the question is such that it is satisfied without the aid of any specific restriction to the law, except that it should be one which very rapidly diminishes as the distance increases, and is insensible at distances appreciable by our senses. Hence, we know that Laplace in his *Mécanique Céleste,* and Poisson after him, have not cared to assume any particular law of force, and even if they had, no means would have been found for its verification. One result of this fact appears to be, that the circumstance of an active force of this nature being sufficient to explain a phenomenon totally different in character from those of cohesion and combination, by which it is obviously suggested, induced Laplace himself to the belief that this was the ultimate law. If such be not the case, I am unable to account for his adoption of such a law, absurd as it appears, in his explanation of the phenomena of heat. It would have been supposed, that this was an opportunity of applying the beautiful analysis of the former parts of his work to the reduction of the molecular law to some simple form. But such is not the case, nor does Poisson, even

in the Memoir where he detects the insufficiency of Laplace's hypothesis of capillary attraction, attempt to ascend higher in the investigation. In his Theory of Heat too, he introduces discrete molecules only for the purpose of generalizing the problems of conduction and radiation, without attempting to solve those of expansion and crystallization: so that he makes no progress whatever in the explanation of phenomena.

3. In my Memoir on Dispersion, I endeavoured to shew that the law of the inverse square of the distance, is that of the attraction or repulsion of the particles of light, and in subsequent Memoirs, I have endeavoured to reduce some of the phenomena of sound and heat to the same law. Nothing, however, was effected with respect to the equilibrium of the molecules. The latter object has lately been accomplished, at least partially, by M. Mossotti, in a Memoir "On the Forces which regulate the Internal Constitution of Bodies." The hypothesis of Mossotti is the same as Dr Knight's, except that the forces vary inversely as the square of the distance. It is proved, that one set of particles may have an atmosphere of another set, the density of which varies rapidly in receding from the surface of the former. M. Mossotti then endeavours to find the conditions of equilibrium of a particle of the first or the material set. It is on this point that I conceive M. Mossotti's hypothesis completely fails. The law of action of two particles as deduced by M. Mossotti, is composed of two parts, a repulsive part which vanishes when the distance is sensible, and an attractive part which varies inversely as the square of the distance. Now when it is borne in mind that the whole set of forces acting on any particle must be sufficient to retain that particle in equilibrium at a certain distance from the one next to it, we shall perceive that this law of action requires that the mutual distance, or the density of the particles, should vary as the magnitude of the body. I do not mean to assert, that the density should be increased in the same ratio as the mass is increased, but that it must be so increased, that the repulsive force of the adjacent particles should be very nearly in the proportion of the linear magnitude of the body. I cannot think this a probable, hardly a possible, condition of matter. The state of the surface may depend, and probably does so,

on the thickness of the solid, provided the solid be very thin, but it can scarcely be conceived to do so in other cases, much less to vary equally with the *superficial extent* of the surface itself.

4. In order then to determine as nearly as possible, what is the law of distribution of the particles of caloric, or the universally diffused system of particles, as well as what is the law of aggregation of the material particles, which determines whether the arrangement have the properties of elasticity, fluidity, solidity, crystalline arrangements, &c. I have examined a number of different arrangements, and investigated the conditions of their equilibrium and stability.

In the present part, I have said little about the application to different states of consistence, deeming it more prudent to make a series of calculations in the first place. In fact, it is most probable that the forms of the results will in all cases, as they certainly are in those I have already tried, be very different from those which a popular view of the subject would suggest. In my treatise on Heat, however, will be found some applications roughly stated, which I hope more fully to investigate in the sequel.

Investigation of the Conditions of Equilibrium.

5. I purpose to commence my investigation, by retaining M. Mossotti's hypothesis of two systems of particles repulsive towards atoms of their own kind, but each respectively attractive towards the atoms of the other. We will call one system of particles *caloric*, and the other *matter;* the masses of the atoms of the former being very small compared with those of the latter. We will suppose the former distributed through space, whilst the latter occupy only certain given positions: in both, the density at different points will be essentially different, but the particles of the latter medium, will in all cases be supposed wherever they exist, to be much more widely separated than those of the former, so that a material particle may be considered as a nucleus, about which the particles of caloric are collected, so as to form its atmosphere.

To find the conditions of equilibrium of a particle of caloric.

6. Let x, y, z be the co-ordinates of a particle of caloric measured from any point as origin, x', y', z' those of another particle, D the density of the caloric estimated by the number of particles in a given volume in the neighbourhood of the former particle: D' the corresponding quantity for the latter; let also X, Y, Z be the co-ordinates of a particle of matter supposed spherical, and collected at its centre of gravity in all cases in which its own attraction or repulsion is to be calculated; call P the mass of an atom of caloric, M that of an atom of matter, each estimated by the attraction or repulsion exerted by it on a unit of either caloric or matter at the distance unity: let V be the sum of each particle of caloric, divided by its distance from that whose co-ordinates are x, y, z; U the sum of each particle of matter divided by its distance from the same point; also, let r be the former distance, R the latter corresponding to the particles respectively, whose co-ordinates are x', y', z'; X, Y, Z, then

$$V = P\iiint \frac{dx'\,dy'\,dz'.D'}{r},$$

$$U = \Sigma\frac{M}{R}.$$

I have adopted integrals for the caloric, as it is supposed that the particles are so near each other, that the variation of action due to the *situation* of a particle with respect to those immediately surrounding it, forms no important element in the calculation. I shall have occasion to mention this subject more explicitly in the sequel.

In order to fix the ideas, let it be supposed that x', y', z'; X. Y, Z are in advance of xyz, so that

$$r = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2},$$

$$R = \sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2};$$

then the action of the caloric on the particle in question parallel to the axis of x is $P.\frac{dV}{dx}$, and since the force is repulsive, it tends to diminish x; for a like reason, that of the matter on the same particle is $P.\frac{dU}{dx}$ tending to increase x; consequently the whole force with which the particle is urged in the direction of the axis of x is $P\left(\frac{dU}{dx}-\frac{dV}{dx}\right)$.

7. By the substitution of integrals in the place of sums, the expression V, as before noticed, is no longer the total action of the caloric on the particle, subject as it is to the powerful variations of action of these particles by which it is immediately surrounded; it is in fact, the total action, omitting these and corresponding variations for the other particles. In order to obtain the conditions of equilibrium of the particle, we must apply another force, viz. the variation of action due to the *place* of the particle.

Without entering into calculation respecting this force, it is evident at once, that its value is increased in the same ratio as the increment of the density at that point, and must consequently vary as $\frac{dD}{dx}$; but whether it might not also vary as D, does not appear so obvious. The following investigation is perhaps more satisfactory.

8. Conceive a portion of the mass to form a prism, of which the axis is parallel to x. Let its section be unity, and its length δx, and suppose the caloric within it to have the uniform density D, then the action on it, due to the above forces, is

$$PD\delta x\left(\frac{dU}{dx}-\frac{dV}{dx}\right):$$

let p be the pressure on the end next the origin, $p+\frac{dp}{dx}\delta x+\&c.$ that on the further end, then we must have

$$\frac{dp}{dx}=PD\left(\frac{dU}{dx}-\frac{dV}{dx}\right);$$

here, then, by taking the aggregate of a large nnmber of particles, we eliminate the effect of the molecular variations which retain any individual one in its place, and may consider p as the actual pressure exerted, by whatever means it matters not, to retain the particles which form one end of the prism in their places. Now the surrounding particles will produce this effect, and it is obvious that the action on any individual particle will vary as the number of particles which act on it, supposing the positions left out of consideration. Thus, suppose (u) particles occupying certain positions to exert a force F, then if two particles could be supposed to occupy the place of each one, the force would become $2F$, and so on. Under these circumstances, then, the repulsion on an individual particle would vary as the density, and whatever be the mode of arrangement, the same law appears the most simple and probable. Similar reasoning applies to the density of the particles *acted on*, and we conclude that $p \propto D^2$.

$$\text{Let } p = \tfrac{1}{2} c D^2;$$

$$\therefore \frac{dp}{dx} = c D \frac{dD}{dx};$$

$$\text{and then } \frac{dD}{dx} = \frac{P}{c}\left(\frac{dU}{dx} - \frac{dV}{dx}\right), \qquad (1);$$

$$\therefore \frac{d^2D}{dx^2} = \frac{P}{c}\left(\frac{d^2U}{dx^2} - \frac{d^2V}{dx^2}\right),$$

$$\frac{d^2D}{dy^2} = \frac{P}{c}\left(\frac{d^2U}{dy^2} - \frac{d^2V}{dy^2}\right),$$

$$\frac{d^2D}{dz^2} = \frac{P}{c}\left(\frac{d^2U}{dz^2} - \frac{d^2V}{dz^2}\right);$$

$$\text{but } \left.\begin{aligned} \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} &= -4\pi P D \qquad (2). \\ \frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2} &= 0 \qquad (3). \end{aligned}\right\},$$

$$\text{hence } \frac{d^2D}{dx^2} + \frac{d^2D}{dy^2} + \frac{d^2D}{dz^2} = -\frac{P}{c}\left(\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}\right)$$

$$= \frac{4\pi P^2}{c} D$$

$$= a^2 D, \qquad (4).$$

if we designate $\frac{4\pi P^2}{c}$ by a^2.

10. The solution of this equation will be found in various Memoirs of M. Poisson and others: it is

$$D = \Sigma \frac{A e^{-a\sqrt{(x_{,}-x)^2+(y_{,}-y)^2+(z_{,}-z)^2}}}{\sqrt{(x_{,}-x)^2+(y_{,}-y)^2+(z_{,}-z)^2}},$$

the symbol Σ having reference to points whose co-ordinates are $x_{,}$, $y_{,}$, $z_{,}$.

With respect to the points in question there can be no difficulty, for, from the form of the solution, it is evident that the medium is influenced symmetrically with respect to any such points, and moreover, the solution of (2) will give V a function of the same quantities, whence equation (1) will determine U to be a function of the same; but the value of U being $\Sigma \frac{M}{R}$ it is obvious that all the quantities are functions only of R, or of $\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}$, hence the value of D becomes

$$D = \Sigma \frac{A e^{-a\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}$$

$$= \Sigma \frac{A e^{-aR}}{R}.$$

11. It may be remarked, that in this solution, as in the corresponding one in my treatise on Heat, I have departed slightly from M. Mossotti's Memoir, by considering the attraction or repulsion of a particle on another, to be proportional to the product of their masses, so that P^2, MP, M^2 are respectively the forces exerted at the distance unity, by P acting on P, P acting on M, and M acting on M. The reasoning of M. Laplace, to which M. Mossotti refers for the proof of the theorem that the pressure varies as the square of the density, I have not retained, as I conceive it does not take notice of the real point of difficulty; namely, that the force which constitutes the right-hand side of the equation, is not the force on any individual particle, unless the sums are so expressed as to indicate the small variations due to the rapid change of action of those particles immediately surrounding that acted on. Indeed, were they the real actions, their sum would

doubtless be zero. This difficulty will be surmounted by taking U and V, as I have done, not for the absolute forces, but for the sum of all the forces if the particles were distributed symmetrically: the pressure, in this case, which arises from the action of the particles contiguous to that under consideration, will obviously vary as the square of the density.

12. I proceed in the next place, to determine the value of V, by two different methods, the comparison of which, will prove that no inaccuracy arises from the adoption of direct variation in deducing the equation (2), provided such variation is known to be uninterrupted.

To integrate equation (2) we must observe that

$$\frac{d^2\Sigma\frac{B}{R}}{dx^2}+\frac{d^2\Sigma\frac{B}{R}}{dy^2}+\frac{d^2\Sigma\frac{B}{R}}{dz^2}=0,$$

and must consequently include $\Sigma\frac{B}{R}$ in the value of V in the place and with the interpretation of an arbitrary constant.

We have $\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=-4\pi P\Sigma\frac{Ae^{-aR}}{R}$.

The complete value of V will therefore be

$$V=\Sigma\frac{B'}{R}-C\Sigma\frac{Ae^{-aR}}{R},$$

C being given by the equation

$$Ca^2=4\pi P;$$

$$\therefore V=\Sigma\left\{\frac{B}{R}-\frac{Ae^{-aR}}{R}\right\}.\frac{4\pi P}{a^2}.$$

13. In order to calculate the value of V directly, it is most convenient to employ polar co-ordinates; the particle M being the pole. Let A be the place of the particle acted on, P that of any other particle, $AP=r$, $AM=R$, angle $PMA=\theta$, $MP=\rho$; radius of a particle $=l$, then $2\pi\rho^2\sin\theta\,d\theta\,d\rho$ is the volume of an elementary annulus;

$$\therefore\ V = \Sigma \iint \frac{2\pi\rho^2 \sin\theta\, d\theta d\rho e^{-a\rho}}{\rho r} . A . P$$

$$= \Sigma \iint \frac{2\pi\rho \sin\theta\, e^{-a\rho} d\theta\, d\rho}{\sqrt{\rho^2 + R^2 - 2\rho R \cos\theta}} . A . P$$

$$= \Sigma AP \int \left(\frac{2\pi e^{-a\rho} \sqrt{\rho^2 + R^2 - 2R\rho\cos\theta}}{R} + C \right) d\rho.$$

For the portion included within a spherical surface, whose centre is M and radius MA, we must have the limits $\theta = 0$, $\theta = \pi$ and ρ less than R; for the remainder ρ is greater than R;

$$\text{hence}\ V = \Sigma AP . \int_l^R \frac{2\pi}{R} d\rho e^{-a\rho} \{R + \rho - (R - \rho)\}$$

$$+ \Sigma AP . \int_R^\infty \frac{2\pi}{R} d\rho e^{-a\rho} \{\rho + R - (\rho - R)\}$$

$$= \Sigma AP . \int_l^R \frac{4\pi\rho e^{-a\rho} d\rho}{R} + \Sigma AP . \int_R^\infty 4\pi e^{-a\rho} d\rho$$

$$= \Sigma AP . \left\{ \frac{4\pi e^{-al}}{R} . \frac{(1 + al)}{a^2} - \frac{4\pi}{a^2} . \frac{e^{-aR}}{R} \right\}$$

$$= \frac{4\pi}{a^2} . P \Sigma A \left(e^{-al} . \frac{1 + al}{R} - \frac{e^{-aR}}{R} \right).$$

The coefficient $\frac{4\pi P}{a^2}$ being the same as obtained by the other method, shews that equation (2) is correct; we also perceive that

$$B = Ae^{-al}(1 + al).$$

14. The equation (1) will give a relation between a, c and the other quantities which we proceed to investigate.

From the value of D (10),

$$\frac{dD}{dx} = \Sigma \frac{A}{R^3} (e^{-aR} + aRe^{-aR}) (X - x),$$

$$\text{also}\ \frac{dV}{dx} = \frac{4\pi P}{a^2} \Sigma \frac{1}{R^3} (B - Ae^{-aR} - AaRe^{-aR}) (X - x),$$

$$\frac{dU}{dx} = M \Sigma \frac{X - x}{R^3};$$

hence the equation gives

$$\Sigma \frac{A}{R^3}(e^{-aR} + aRe^{-aR})(X - x)$$

$$= \frac{P}{c} . \Sigma \left\{\frac{M}{R^3} - \frac{4\pi P}{a^2 R^3} . \overline{B - A(e^{-aR} + aRe^{-aR})}\right\}(X - x),$$

whence it evidently follows that

$$\Sigma \left(\frac{M}{R^3} - \frac{4\pi PB}{a^2 R^3}\right) = 0,$$

$$\text{and } \frac{4\pi P^2}{ca^2} = 1.$$

The last equation merely verifies the operation, since the value of a which it gives, is no other than its assumed value in (Art. 9).

The other equation gives $M = \dfrac{4\pi PB}{a^2}$

$$= \frac{cB}{P}.$$

$$c = \frac{PM}{B};$$

but from the nature of c, it evidently varies as P^2, call it therefore aP^2, where a is a quantity independent both of M and P; the result is

$$a^2 = \frac{4\pi}{a},$$

$$B = \frac{M}{aP},$$

or a is the same for all substances, whilst B varies as the attractive energy of the particle of matter.

15. This conclusion is of great importance, as it enables us to calculate the effect of any individual particle independently of those by which it is accompanied. In fact, whatever be the nature of the mass, any individual particle will be surrounded by an atmosphere of caloric, the density of which varies as $\dfrac{e^{-aR}}{R}$, where R is the distance from its

centre; whilst the density at a given distance varies only as the attractive energy of the particle. Of course, the expression *density* does not signify the actual amount of aggregation of particles, but merely the aggregation so far as it depends on the particle under consideration.

16. We may verify our conclusion with respect to the value of c, by the following method:

Conceive only one particle to exist. At a considerable distance R from its centre, the principal forces which act on a particle of its surrounding caloric, are the attraction of the particle and the repulsion of the caloric.

The former force is $\frac{MP}{R^2}$.

The latter $\frac{4\pi P^2 A}{R^2} . \int e^{-ar} r dr = \frac{4\pi P^2 A}{R^2} \{C - \frac{r}{a} e^{-ar} - \frac{1}{a^2} e^{-ar}\}$,

the value of which from $r = l$ to $r = a$ *considerable quantity* is very nearly

$$\frac{4\pi P^2 A}{R^2 a^2} e^{-al} (1 + al) = \frac{4\pi P^2 B}{R^2 a^2};$$

$$\text{hence } \frac{4\pi P^2 B}{R^2 a^2} = \frac{MP}{R^2},$$

$$a^2 = \frac{4\pi PB}{M},$$

the same value as we obtained by the former process.

17. We have hitherto omitted any consideration of a uniform layer of caloric distributed over space, so as to act equally on every point. It is clear, that the effect of such caloric will be found by retaining D as the excess of density above this uniform density q. The correct value of V will now be found by subtracting from its value above the sum of every mass displaced by a material molecule divided by its distance from the point under consideration; hence, all we have to do is to diminish V by a quantity

$$= \Sigma q . \frac{4\pi}{3} l^3 \frac{P}{R} = \frac{4\pi}{3} q l^3 P \Sigma . \frac{1}{R};$$

$$\therefore\ V = \frac{4\pi P}{a^2} \Sigma A \left\{ e^{-al} \frac{(1 + al)}{R} - \frac{e^{-aR}}{R} \right\} - \frac{4\pi}{3} q l^3 P \Sigma \frac{1}{R},$$

and $B = A e^{-al}(1 + al) - \dfrac{a^2 q l^3}{3}$;

$$\therefore\ A = B \cdot \frac{e^{al}}{1 + al} + \frac{a^2 q l^3}{3} \cdot \frac{e^{al}}{1 + al}.$$

The equations in (14) are not affected by this consideration, consequently B is independent of the mean density; and A is increased proportionally to it.

18. I propose next to determine the mutual action of two particles.

We have seen that the atmosphere of any particle is perfectly independent of that of the surrounding particles: it follows, that the action of two particles on each other, is also independent of the surrounding medium. The latter supposes, however, that the *pressure* which is exerted by the caloric is due to the actions of particles so arranged as to produce equilibrium; in fact, the pressure on the surface of a material particle A, even as far only as it depends on the caloric which constitutes the atmosphere of B, will vary with the attractions of the other particles on it, except the system be in equilibrium, in which case we may suppose, as we have already done, that the pressure corresponding to the density $D \propto D$

$$= hD.$$

19. Our first point will be to find the value of this pressure.

Let a be the distance between the centres of the particles, l their radius, P any point in the particle on which the pressure is to be determined; then the area of an annulus is $2\pi l^2 \sin\theta d\theta$,

and the pressure on it $2\pi l^2 h A \dfrac{e^{-aR}}{R} \sin\theta d\theta$;

hence, the resolved part of the whole pressure in the direction of the line joining the centres of the particles, is

$$2\pi l^2 hA\int\frac{e^{-\alpha\sqrt{a^2+l^2-2al\cos\theta}}.\sin\theta\cos\theta}{\sqrt{a^2+l^2-2al\cos\theta}}.d\theta = Q,$$

$$\text{but } \frac{\sin\theta d\theta}{\sqrt{a^2+l^2-2al\cos\theta}} = \frac{dR}{al},$$

$$\therefore\ Q = \frac{2\pi lhA}{a}\int\frac{e^{-\alpha R}dR\,(a^2+l^2-R^2)}{2al}$$

$$= \frac{\pi hA}{a^2}\int(a^2+l^2-R^2)\,e^{-\alpha R}dR,$$

the limits being $R = a - l$, and $R = a + l$;

$$\text{hence } Q = \frac{2\pi hA}{a^2}\left\{\left(\frac{al}{\alpha}+\frac{a+l}{\alpha^2}+\frac{1}{\alpha^3}\right)e^{-\alpha(a+l)} - \left(-\frac{al}{\alpha}+\frac{a-l}{\alpha^2}+\frac{1}{\alpha^3}\right)e^{-\alpha(a-l)}\right\}.$$

The attraction of the caloric is (16 and 17) very nearly,

$$\frac{4\pi MPA}{a^2}\left\{\frac{1}{\alpha^2}e^{-\alpha l}+\frac{l}{\alpha}e^{-\alpha l}-\frac{1}{\alpha^2}e^{-\alpha a}-\frac{a}{\alpha}e^{-\alpha a}\right\} - \frac{4\pi}{3}\frac{ql^3MP}{a^2},$$

whilst the mutual repulsion of the two particles is $\frac{M^2}{a^2}$;

hence, the expression for the whole force of mutual *attraction* of the particles towards each other, is

$$S = \frac{4\pi MPA}{a^2}\left\{\frac{e^{-\alpha l}}{\alpha^2}+\frac{l}{\alpha}e^{-\alpha l}-\frac{1}{\alpha^2}e^{-\alpha a}-\frac{a}{\alpha}e^{-\alpha a}\right\} - \frac{M^2}{a^2} - \frac{4\pi}{3}\frac{ql^3MP}{a^2}$$

$$-\frac{2\pi hA}{a^2}\left\{\left(\frac{al}{\alpha}+\frac{a+l}{\alpha^2}+\frac{1}{\alpha^3}\right)e^{-\alpha(a+l)} - \left(-\frac{al}{\alpha}+\frac{a-l}{\alpha^2}+\frac{1}{\alpha^3}\right)e^{-\alpha(a-l)}\right\}.$$

20. Here we have not taken into consideration the circumstance that the mass of the particle will not be acted on exactly as if collected at its centre of gravity. It has been supposed that it is so collected, and that the caloric then extends to infinity, so that the attraction is due to a quantity of caloric lying in a sphere about the attracting particle at the distance of the attracted one. Now, in fact, nearly one half the attracted particle will not be acted on so much by the laminæ beyond its surface, whilst the other portion is actually acted on by particles beyond the laminæ at the centre; but as the density of the

former laminæ is greater, and the part of the body on which it acts less, we cannot have erred much in taking the mean as the correct value of the attraction.

Indeed, if there be any error committed, it is obvious that we have estimated the attraction too high; both from the greater density being that which we have supposed to have full agency, and from the fact, that the actual attraction on parts lying at a distance from the centre, is not in the direction of the line joining the centres of the particles. It may then be conceived, that the above expression is rather too great for the attraction, and it will appear presently that its value even as I have given it, is negative.

For we have already proved (14) that $\frac{4\pi PB}{a^2} = M$;

$$\therefore \frac{4\pi PA}{a^2}(1 + al)\, e^{-al} - \frac{4\pi P l^3 q}{3} = M \, (17);$$

hence

$$S = -\frac{4\pi MPA}{a^2 \alpha^2}(1 + \alpha a)\, e^{-\alpha a} - \frac{2\pi h A}{a^2 \alpha^2} e^{-\alpha a} \tfrac{2}{3} a^2 l^3 (1 + \alpha a),$$

an essentially negative result.

21. We may however introduce a positive quantity into this expression, by conceiving each molecule as a compound one of two different kinds of particles attracting each other, as we proceed to shew.

$$\text{Let } U' = \Sigma \frac{M'}{R'},$$

then the action on a particle of caloric is

$$P\left(\frac{dU}{dx} + \frac{dU'}{dx} - \frac{dV}{dx}\right);$$

hence all the equations for motion are unaffected:

$$\text{and } D = \Sigma \frac{Ae^{-\alpha R}}{R} + \Sigma \frac{A' e^{-\alpha R'}}{R'},$$

$$V = \Sigma\left\{\frac{B}{R} - \frac{Ae^{-\alpha R}}{R} + \frac{B'}{R'} - \frac{A' e^{-\alpha R'}}{R'}\right\} \frac{4\pi P}{a^2};$$

hence the equation, $\frac{dD}{dx} = \frac{P}{c}\left(\frac{dU}{dx} + \frac{dU'}{dx} - \frac{dV}{dx}\right)$

$$\text{gives } \Sigma\frac{A}{R^3}(e^{-aR} + aRe^{-aR}) + \Sigma\frac{A'}{R'^3}(e^{-aR'} + aRe^{-aR'})$$

$$= \frac{P}{c}\Sigma\left[\frac{M}{R^3} + \frac{M'}{R'^3} - \frac{4\pi P}{a^2}\left\{\frac{B}{R^3} + \frac{B'}{R'^3} - A(e^{-aR} + aRe^{-aR}) - A'(e^{-aR'} + aR'e^{-aR})\right\}\right],$$

$$\text{hence } M + M' = \frac{4\pi P(B + B')}{a^2}.$$

The expression for the attraction of a particle of the first substance on one of the same kind, whose mutual distance is a, is

$$S = \frac{4\pi MPA}{a^2}\left\{\frac{(1+al)e^{-al}}{a^2} - \frac{(1+aa)e^{-aa}}{a^2}\right\} - \frac{M^2}{a^2}$$

$$- \frac{2\pi hA}{a^2}e^{-aa}\left\{\left(\frac{a+l+aal}{a^2} + \frac{1}{a^3}\right)e^{-al} - \left(\frac{a-l-aal}{a^2} + \frac{1}{a^3}\right)e^{al}\right\};$$

and a similar expression, only accenting the letters, is true for the mutual attractions of the other *similar* particles; call it S': also, since M attracts M', if T be the attraction of M' in virtue of M, we shall have

$$T = \frac{4\pi PM'A}{a^2}\left\{\frac{(1+al)e^{-al}}{a^2} - \frac{(1+aa)e^{-aa}}{a^2}\right\} + \frac{MM'}{a^2}$$

$$- \frac{2\pi hA}{a^2}e^{-aa}\left\{\left(\frac{a+l'+aal'}{a^2} + \frac{1}{a^3}\right)e^{-al'} - \&c.\right\},$$

$$T' = \frac{4\pi PMA'}{a^2}\left\{\frac{(1+al')e^{-al'}}{a^2} - \frac{(1+aa)e^{-aa}}{a^2}\right\} + \frac{MM'}{a^2}$$

$$- \frac{2\pi hA'}{a^2}e^{-aa}\left\{\left(\frac{a+l+aal}{a^2} + \frac{1}{a^3}\right)e^{-al} - \&c.\right\};$$

hence, the whole mutual attraction of a compound particle is

$$S + S' + T + T'$$

$$= \frac{4\pi P}{\alpha^2 a^2}\{BM + B'M' - (1+\alpha a)e^{-\alpha a}(AM + A'M')\} - \frac{(M-M')^2}{a^2}$$

$$- \frac{2\pi h}{\alpha^2 a^2} e^{-\alpha a}\left[\{A + A'\}\left\{e^{-\alpha l}\left(a + l + \alpha a l + \frac{1}{\alpha}\right) - e^{+\alpha l}.\left(a - l - \alpha a l + \frac{1}{\alpha}\right)\right.\right.$$

$$\left.\left. + e^{-\alpha l'}(a + l') + \ldots\ldots\ldots\ldots\right\}\right].$$

Now for a single particle $M = \frac{4\pi BP}{\alpha^2}$, and there is no reason to suppose B different in other cases; hence, the mutual attraction of the compound particles is

$$\frac{2MM'}{a^2} - \frac{M}{Ba^2}(AM + A'M')(1 + \alpha a)e^{-\alpha a}$$

$$- \frac{Mh}{2PBa^2}e^{-\alpha a}(A + A')\{e^{-\alpha l}.(a + l) + \ldots\ldots\ldots\ldots\}.$$

Now if they are at a distance from each other, the quantity $e^{-\alpha a}$ is very small, and the force is $\frac{2MM'}{a^2}$ varying inversely as the square of the distance, which is the known law of gravitation.

22. I shall not dwell longer on this point, as the difficulty is not to obtain a portion of the expression which shall vary inversely as the square of the distance; for this will be at once accomplished either by the above method, or by supposing the attraction of M on P a little greater than MP, as M. Mossotti has done, or by taking into the calculation the caloric which is displaced by a particle, either by the one attracting, or that acted on, which in accuracy ought to be done. But the difficulty is to obtain an expression for the mutual action of two particles, which shall express those facts of Boscovich's hypothesis specified in the Introduction, and which are clearly essential to the nature of a molecular action.

To accomplish this object, I have supposed *all* the particles repulsive; which hypothesis requires that the density of the caloric within the medium, be less than that without. I shall not attempt to justify this hypothesis, or to prove that its apparent complexity, as compared with the received one, affords a strong *à priori* argument against its correctness. The only way to obtain final accuracy, is to subject to rigid calculation any hypothesis which may suggest itself, and to retain that which gives results consistent with facts. And should it be found that a little difficulty attaches itself to the one in question, we may expect either that the difficulty itself will vanish, or the hypothesis will be found unnecessary from after-attention to a more simple one. I may state, that I have spent a considerable portion of time in trying other hypotheses, but at present can find none which so apparently coincides with known phenomena as that which I have just stated.

23. Let us then determine the conditions of equilibrium of a system in which the atoms of caloric are repulsive to those of matter.

Assume the density of the external caloric to be q, and that of the internal q', so that by writing $q - q'$ for D, we may adapt some of our previous investigations to this case.

We have $\frac{dp}{dx} = -Pq'\left(\frac{dU}{dx} + \frac{dV}{dx}\right)$,

where V has reference to every particle.

But $p = \frac{1}{2}cq'^2$;

$$\therefore \frac{dp}{dx} = cq'\frac{dq'}{dx},$$

$$\frac{dq'}{dx} = -\frac{P}{c}\left(\frac{dU}{dx} + \frac{dV}{dx}\right);$$

$$\text{and } \frac{dD}{dx} = \frac{dq}{dx} - \frac{dq'}{dx} = \frac{P}{c}\left(\frac{dU}{dx} + \frac{dV}{dx}\right) + \frac{dq}{dx}.$$

Now if there were no material particle, we should have

$$\frac{dq}{dx} = -\frac{P}{c}\frac{dV_{,}}{dx},$$

where $V_{,}$ is the quantity which V becomes for a homogeneous medium of density q; if then we assume $V_{,} - V = V'$; where V' is the function due to a mass of particles equal to those displaced by the repulsion, and situated in the places from which they have been driven off; we shall get

$$\frac{dD}{dx} = \frac{P}{c}\left(\frac{dU}{dx} - \frac{dV'}{dx}\right);$$

$$\therefore \frac{d^2D}{dx^2} = \frac{P}{c}\left(\frac{d^2U}{dx^2} - \frac{d^2V'}{dx^2}\right);$$

$$\text{and } \frac{d^2D}{dx^2} + \frac{d^2D}{dy^2} + \frac{d^2D}{dz^2} = -\frac{P}{c}\left(\frac{d^2V'}{dx^2} + \frac{d^2V'}{dy^2} + \frac{d^2V'}{dz^2}\right);$$

$$\text{but } \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -4\pi Pq',$$

$$\frac{d^2V_{\prime}}{dx^2} + \frac{d^2V_{\prime}}{dy^2} + \frac{d^2V_{\prime}}{dz^2} = -4\pi Pq;$$

$$\therefore \frac{d^2(V_{\prime}-V)}{dx^2} + \frac{d^2(V_{\prime}-V)}{dy^2} + \frac{d^2(V_{\prime}-V)}{dz^2} = -4\pi P(q-q'),$$

$$\text{or } \frac{d^2V'}{dx^2} + \frac{d^2V'}{dy^2} + \frac{d^2V'}{dz^2} = -4\pi PD;$$

$$\therefore \frac{d^2D}{dx^2} + \frac{d^2D}{dy^2} + \frac{d^2D}{dz^2} = \frac{4\pi P^2}{c}.D,$$

$$= a^2D;$$

the solution of which equation is

$$D = \frac{Ae^{-aR}}{R};$$

and, as in the former case, it evidently follows that

$$V' = \Sigma\left(\frac{B}{R} - \frac{Ae^{-aR}}{R}\right).\frac{4\pi P}{a^2}.$$

24. By employing a process precisely analogous to that in (13), we obtain the value of V' directly, taking into the account the caloric displaced by the material particles; the expression is

$$V' = \frac{4\pi P}{a^2}\Sigma A\left\{\frac{e^{-al}(1+al)}{R} - \frac{e^{-aR}}{R}\right) + \frac{4\pi}{3}l^3qP\Sigma\frac{1}{R};$$

$$\text{hence } B = Ae^{-\alpha l}(1+\alpha l) + \frac{\alpha^2 l^3 q}{3},$$

and we obtain also, as in (14),

$$M = \frac{cB}{P} = \frac{4\pi PB}{\alpha^2},$$

$$(1+\alpha l)\,Ae^{-\alpha l} = \frac{\alpha^2 M}{4\pi P} - \frac{\alpha^2 l^3 q}{3}$$

$$= \frac{\alpha^2}{4\pi P}\left(M - \frac{4\pi l^3 Pq}{3}\right),$$

$$\frac{4\pi AMPe^{-\alpha l}(1+\alpha l)}{\alpha^2 a^2} = \frac{M^2}{a^2} - \frac{4\pi Ml^3 Pq}{3a^2};$$

which expressions will simplify that for the force of two material particles on each other, by striking out several identical terms.

To find the mutual action of two particles of matter together with the caloric surrounding them, on the hypothesis that matter is repulsive towards caloric.

25. Since the caloric surrounding the particle A, whose action on B we are about to estimate, is diminished by A's repulsion, the external mass will no longer produce an effect equal in all directions, whose actual value is therefore zero; but will exert a force on B equivalent to the *attraction* of a mass similar, and similarly situated to the mass displaced.

The set of forces, then, which act on B through the means of A, are

(1). The repulsion of A on B.

(2). The attraction of a mass of caloric equal to that displaced by the volume of A.

(3). The attraction of a mass of caloric equal to that displaced by the repulsion of A, and

(4). The pressure on the surface of B resolved in one direction along the line joining the centres of A, B.

26. If a be the distance between the centres of A and B, the expression for their repulsion is $\frac{M^2}{a^2}$, which is the first force.

27. The value of the second force is also $\frac{4\pi l^3}{3a^2}.qMP$, calling the exterior or mean density q.

28. To obtain the third force, we must divide the displaced caloric, or rather a portion equal to it placed in a position directly opposite, into two portions; the one containing all that is included in a sphere whose centre is the centre of A, and radius the distance to that point of B which is nearest to A; the other, the portion arising from the spherical shell included between two surfaces to radii equal to the distances of the nearest and most distant point of B, from the centre of A.

The former of these is easily found as in (16), equal to

$$\frac{4\pi MPA}{a^2\alpha^2}\left[e^{-\alpha l}(1+\alpha l) - e^{-\alpha(a-l)}\{1+\alpha(a-l)\}\right].$$

To obtain the latter, we will first omit the consideration of the portion which would occupy the place of B, supposing that particle removed, and consequently take no notice of the quantity which ought to be displaced there; by this means, it is obvious that we shall estimate the attraction a little too highly; and we shall see that the portion, taken as we have supposed, is actually less than would be obtained by conceiving the mass of B collected at its centre; consequently the whole attraction is considerably less than that given in (20). Now we saw that the resultant action even on that calculation was essentially negative, it appears then that a more rigorous analysis increases rather than diminishes the difficulty attendant on an *attractive* atmosphere of caloric.

29. Let us then proceed to the calculation.

The action of a mass of caloric in a spherical shell of thickness δR, whose radius is R on a similar portion of a shell of the body B at radius ρ, is easily seen to be*

$$\frac{AMP}{V}.\frac{4\pi R^2\delta R e^{-\alpha R}}{R}.(\iint 2\pi \sin\phi\, d\rho\, d\phi \cos\phi);$$

* For the construction of the figure, &c. see the Note (*a*) at the end.

of which the last factor

$$= \int\left(-\frac{\pi d\rho}{2}\cos 2\phi + C\right)$$

$$= \int\frac{\pi}{2}d\rho\,(1 - \cos 2\phi) = \pi d\rho \sin^2\phi$$

$$= \int \pi d\rho \left\{1 - \left(\frac{a^2+\rho^2-l^2}{2a\rho}\right)^2\right\}$$

$$= \pi\left\{\rho - \int\frac{(a^2-l^2)^2 + 2\rho^2(a^2-l^2) + \rho^4}{4a^2\rho^2}d\rho\right\}$$

$$= \pi\left[\rho - \frac{1}{4a^2}\left\{-\frac{(a^2-l^2)^2}{\rho} + 2(a^2-l^2)\rho + \frac{\rho^3}{3}\right\}\right]$$

$$= \pi\left[a + l - R - \frac{1}{4a^2}\left\{(a^2-l^2)^2\left(\frac{1}{R} - \frac{1}{a+l}\right) + 2(a^2-l^2)(a+l-R) + \frac{(a+l)^3 - R^3}{3}\right\}\right];$$

consequently the whole attraction is

$$\frac{AMP4\pi^2}{V}\int dR e^{-\alpha R}\left\{(a+l-R)R - \frac{1}{4a^2}\left(\overline{a+l}\,\overline{a-l}^2\,.\,\overline{a+l-R}\right) - \frac{1}{2a^2}(a^2-l^2)(a+l-R)R - \frac{(a+l)^3-R^3}{12a^2}R\right\}$$

$$= \frac{AMP4\pi^2}{V}\int dR e^{-\alpha R}\left\{-\frac{(a^2-l^2)^2}{4a^2} + \frac{2}{3a^2}(a+l)(a^2-al+l^2)R - \frac{a^2+l^2}{2a^2}R^2 + \frac{R^4}{12a^2}\right\}$$

$$= \frac{AMP}{V}\cdot\frac{4\pi^2e^{-\alpha R}}{a^2}\left\{\frac{(a^2-l^2)^2}{4\alpha} - \frac{2}{3a^2}(a+l)(a^2-al+l^2)\left(\frac{1+\alpha R}{\alpha^2}\right) + \frac{a^2+l^2}{2}\left(\frac{R^2}{\alpha} + \frac{2R}{\alpha^2} + \frac{2}{\alpha^3}\right) - \frac{1}{12}\left(\frac{R^4}{\alpha} + \frac{4R^3}{\alpha^2} + \frac{12R^2}{\alpha^3} + \frac{24R}{\alpha^4} + \frac{24}{\alpha^5}\right)\right\}$$

$$= \frac{AMP}{V} \cdot \frac{4\pi^2 e^{-\alpha a}}{\alpha a^2} \left[e^{-\alpha l} \left\{ \frac{(a^2-l^2)^2}{4} - \frac{2}{3a^2}(a^3+l^3)\, \frac{1+\alpha(a+l)}{\alpha} \right.\right.$$

$$+ \frac{a^2+l^2}{2}\left(\overline{a+l}^2 + \frac{2\,\overline{a+l}}{\alpha} + \frac{2}{\alpha^2}\right) - \frac{1}{12}\left(\overline{a+l}^4 + \frac{4\,\overline{a+l}^3}{\alpha} + \frac{12\,\overline{a+l}^2}{\alpha^2}\right.$$

$$\left.\left. + \frac{24\,\overline{a+l}}{\alpha^3} + \frac{24}{\alpha^4}\right)\right\}$$

$$- e^{+\alpha l} \left\{ \frac{(a^2-l^2)^2}{4} - \frac{2}{3a^2}(a^3+l^3)\left(\frac{1+\alpha\,\overline{a-l}}{4}\right) \right.$$

$$+ \frac{a^2+l^2}{2}\left(\overline{a-l}^2 + \frac{2\,\overline{a-l}}{\alpha} + \frac{2}{\alpha^2}\right) - \frac{1}{12}\left(\overline{a-l}^4 + \frac{4\,\overline{a-l}^3}{\alpha} + \frac{12\,\overline{a-l}^2}{\alpha^2}\right.$$

$$\left.\left.\left. + \frac{24\,\overline{a-l}}{\alpha^3} + \frac{24}{\alpha^4}\right)\right\}\right].$$

The quantity under the bracket following $e^{-\alpha l}$, becomes by addition

$$\frac{a^4 - 2a^2l^2 + l^4}{4} + \frac{a^2+l^2}{2}\left(a^2 + \frac{2a}{\alpha} + \frac{2}{\alpha^2} + 2\,\frac{1+\alpha a}{\alpha}.\,l + l^2\right)$$

$$- \frac{1}{12}\left\{a^4 + \frac{4a^3}{\alpha} + \frac{12a^2}{\alpha^2} + \frac{24a}{\alpha^3} + \frac{2a}{\alpha^4}\right.$$

$$+ \left(4a^3 + \frac{12a^2}{\alpha} + \frac{24a}{\alpha} + \frac{24}{\alpha^3}\right).\,l + \left(6a^2 + \frac{12a}{\alpha} + \frac{12}{\alpha^2}\right).\,l^2$$

$$\left. + \left(4a + \frac{4}{\alpha}\right).\,l^3 + l^4\right\} - \frac{2}{3a^2\alpha}(a^3+l^3)(1+\alpha a+\alpha l)$$

$$= \frac{a^4}{4} + \frac{a^2}{2}\left(a^2 + \frac{2a}{\alpha} + \frac{2}{\alpha^2}\right) - \frac{1}{12}\left(a^4 + \frac{4a^3}{\alpha} + \frac{12a^2}{\alpha^2} + \frac{24a}{\alpha^3} + \frac{24}{\alpha^4}\right)$$

$$+ \left\{a^2\left(\frac{1+\alpha a}{\alpha}\right) - \frac{1}{3}\left(a^3 + \frac{3a^2}{\alpha} + \frac{6a}{\alpha^2} + \frac{6}{\alpha^3}\right)\right\}.\,l$$

$$+ \left\{\frac{a^2}{2} + \frac{1}{2}\left(a^2 + \frac{2a}{\alpha} + \frac{2}{\alpha^2}\right) - \frac{a^2}{2} - \frac{1}{2}\left(a^2 + \frac{2a}{\alpha} + \frac{2}{\alpha^2}\right)\right\}.\,l^2$$

$$+ \left(\frac{1+\alpha a}{\alpha} - \frac{1}{3}\,\frac{1+\alpha a}{\alpha}\right).\,l^3 + \left(\frac{1}{4} + \frac{1}{2} - \frac{1}{12}\right).\,l^4 - \frac{2}{3a^2\alpha}(a^3+l^3)(1+\alpha a+\alpha l)$$

$$= \frac{2}{3}a^4 + \frac{2}{3}\frac{a^3}{\alpha} - \frac{2\,a}{\alpha^3} - \frac{2}{\alpha^4} + \left(\frac{2}{3}a^3 - \frac{2\,a}{\alpha^2} - \frac{2}{\alpha^3}\right) l$$

$$+ \frac{2}{3}\left(\frac{1+\alpha a}{\alpha}\right) l^3 + \frac{2}{3} l^4 - \frac{2}{3}(a^3 + l^3)\left(\frac{1 + \alpha a + \alpha l}{\alpha}\right)$$

$$= \frac{2}{3}\left\{a^4 + \frac{a^3}{\alpha} - \frac{3\,a}{\alpha^3} - \frac{3}{\alpha^4} + \left(a^3 - \frac{3\,a}{\alpha^2} - \frac{3}{\alpha^3}\right).l + \left(\frac{1+\alpha a}{\alpha}\right).l^3 + l^4\right\}$$

$$- \frac{2}{3}(a^3 + l^3)\left(\frac{1+\alpha a + \alpha l}{\alpha}\right)$$

$$= \frac{2}{3}\left(a^4 + \frac{C}{\alpha} + Cl + \frac{1+\alpha a}{\alpha} l^3 + l^4\right) - \frac{2}{3}(a^3 + l^3)\left(\frac{1+\alpha a + \alpha l}{\alpha}\right),$$

if we denote $a^3 - \frac{3\,a}{\alpha^2} - \frac{3}{\alpha^3}$ by C.

By the substitution of this value and the corresponding value of the coefficient of $e^{\alpha l}$, the attraction becomes

$$\frac{AMP\,8\pi^2 e^{-\alpha a}}{3\,V\alpha a^2}\left\{e^{-\alpha l}\left(a^4 + \frac{C}{\alpha} + Cl + \frac{1+\alpha a}{\alpha} l^3 + l^4\right)\right.$$

$$- e^{+\alpha l}\left(a^4 + \frac{C}{\alpha} - Cl - \frac{1+\alpha a}{\alpha} l^3 + l^4\right)$$

$$\left. - (a^3 + l^3)\left(\frac{1+\alpha a + \alpha l}{\alpha} e^{-\alpha l} - \frac{1+\alpha a - \alpha l}{\alpha} e^{\alpha l}\right)\right\}.$$

30. I proceed next to find the value of the term omitted, by taking the mass of displaced caloric between limits involving B itself. It is obvious, that we have calculated the attraction of a mass which does not exist, and shall have to subtract the value which we obtain in order to get the correct attraction.

Now we have to estimate the resolved part, along the line joining the centres of the molecules, of the attraction of a mass of fluid of variable density on the different parts of a solid conceived to occupy the same space with itself.

If we take any element P of the fluid, and estimate its attraction on the whole solid, the result will obviously be the attraction of the solid on this element.

Let ρ be the distance of an element from the centre of the particle B, and retain the remaining notation of the last problem, then will be the volume of an elementary annulus

$$= 2\pi\rho^2\, d\rho\, d\theta \sin\theta,$$

and the mass attracting this, is

$$\frac{M\rho^3}{l^3};$$

consequently the resolved part of the attraction in the required direction,

$$= \iint \frac{M\rho^3}{l^3\rho^2} \cdot 2\pi\rho^2\, d\rho\, d\theta \sin\theta \cos\theta \frac{PA \,.\, e^{-ar}}{r}$$

$$= \frac{2\pi AMP}{l^3} \iint \frac{\rho^3 e^{-ar}}{r} \sin\theta \cos\theta\, d\theta\, d\rho.$$

Now $r^2 = a^2 + \rho^2 - 2a\rho\cos\theta$;

$$\therefore\ rdr = a\rho \sin\theta\, d\theta,$$

and the attraction

$$= \frac{2\pi AMP}{al^3} \iint e^{-ar} \rho^2 \cos\theta\, dr\, d\rho$$

$$= \frac{2\pi AMP}{al^3} \iint e^{-ar} \rho^2 \frac{a^2 + \rho^2 - r^2}{2a\rho}\, dr\, d\rho$$

$$= \frac{\pi AMP}{a^2 l^3} \iint e^{-ar} \rho\,(a^2 + \rho^2 - r^2)\, dr\, d\rho$$

$$= \frac{\pi AMP}{a^2 l^3} \int \left\{ C - \frac{(a^2 + \rho^2)}{a} e^{-ar} + \left(\frac{r^2}{a} + \frac{2r}{a^2} + \frac{2}{a^3} \right) e^{-ar} \right\} \rho\, d\rho;$$

the limits being $r = a + \rho$, and $r = a - \rho$

$$= \frac{\pi AMP}{a^2 l^3} \int \rho\, d\rho \left\{ \frac{a^2 + \rho^2}{a} e^{-a(a-\rho)} - \left(\frac{\overline{a-\rho}^2}{a} + \frac{2\,\overline{a-\rho}}{a^2} + \frac{2}{a^3} \right) e^{-a(a-\rho)} \right.$$

$$\left. - \frac{a^2 + \rho^2}{a} e^{-a(a+\rho)} + \left(\frac{(a+\rho)^2}{a} + \frac{2(a+\rho)}{a^2} + \frac{2}{a^3} \right) e^{-a(a+\rho)} \right\}$$

$$= \frac{\pi AMPe^{-\alpha a}}{a^2 l^3} \int_0^l \rho\, d\rho \left\{ \frac{a^2+\rho^2}{\alpha} e^{\alpha\rho} - \left(\frac{a^2 - 2a\rho + \rho^2}{\alpha} + \frac{2\,\overline{a-\rho}}{\alpha^2} + \frac{2}{\alpha^3} \right) e^{\alpha\rho} \right.$$

$$\left. - \frac{a^2+\rho^2}{\alpha} e^{-\alpha\rho} + \left(\frac{(a+\rho)^2}{\alpha} + \frac{2(a+\rho)}{\alpha^2} + \frac{2}{\alpha^3} \right) e^{-\alpha\rho} \right\}$$

$$= \frac{2\pi APMe^{-\alpha a}}{a^2 l^3} \int_0^l \rho\, d\rho \left\{ \left(\frac{a\rho}{\alpha} + \frac{\rho}{\alpha^2} - \frac{a}{\alpha^2} - \frac{1}{\alpha^3} \right) e^{\alpha\rho} \right.$$

$$\left. + \left(\frac{a\rho}{\alpha} + \frac{\rho}{\alpha^2} + \frac{a}{\alpha^2} + \frac{1}{\alpha^3} \right) e^{-\alpha\rho} \right\}$$

$$= \frac{2\pi AMPe^{-\alpha a}}{a^2 l^3} \left[\left\{ \left(\frac{a}{\alpha} + \frac{1}{\alpha^2} \right) \cdot \left(\frac{\rho^2}{\alpha} - \frac{2\rho}{\alpha^2} + \frac{2}{\alpha^3} \right) - \left(\frac{a}{\alpha^2} + \frac{1}{\alpha^3} \right) \cdot \left(\frac{\rho}{\alpha} - \frac{1}{\alpha^2} \right) \right\} e^{+\alpha\rho} \right.$$

$$\left. - \left\{ \left(\frac{a}{\alpha} + \frac{1}{\alpha^2} \right) \cdot \left(\frac{\rho^2}{\alpha} + \frac{2\rho}{\alpha^2} + \frac{2}{\alpha^3} \right) + \left(\frac{a}{\alpha^2} + \frac{1}{\alpha^3} \right) \cdot \left(\frac{\rho}{\alpha} + \frac{1}{\alpha^2} \right) \right\} e^{-\alpha\rho} \right]$$

$$+ C$$

$$= \frac{2\pi AMPe^{-\alpha a}}{a^2 l^3} \left[\left\{ \frac{1+\alpha a}{\alpha^2} \left(\frac{l^2}{\alpha} - \frac{2l}{\alpha^2} + \frac{2}{\alpha^3} \right) - \frac{1+\alpha a}{\alpha^3} \left(\frac{l}{\alpha} - \frac{1}{\alpha^2} \right) \right\} e^{\alpha l} \right.$$

$$\left. - \left\{ \frac{1+\alpha a}{\alpha^2} \left(\frac{l^2}{\alpha} + \frac{2l}{\alpha^2} + \frac{2}{\alpha^3} \right) + \frac{1+\alpha a}{\alpha^3} \left(\frac{l}{\alpha} + \frac{1}{\alpha^2} \right) \right\} e^{-\alpha l} \right]$$

$$= \frac{2\pi AMPe^{-\alpha a}}{\alpha^3 a^2 l^3} (1+\alpha a) \left\{ \left(l^2 - \frac{3l}{\alpha} + \frac{3}{\alpha^2} \right) e^{\alpha l} - \left(l^2 + \frac{3l}{\alpha} + \frac{3}{\alpha^2} \right) e^{-\alpha l} \right\};$$

which must be subtracted from the expression above, in order to give that part of the total action designated by (3).

31. With respect to the fourth part of the total action, it is evident that it differs from the expression already obtained on the previous hypothesis (19) only in sign. Its value is, therefore,

$$\frac{2\pi hA}{a^2 \alpha^3} (1+\alpha a) e^{-\alpha a} \{ (1+\alpha l) e^{-\alpha l} - (1-\alpha l) e^{\alpha l} \},$$

or in the approximate form

$$\frac{4\pi hAe^{-\alpha a}}{3a^2 \alpha^2} (1+\alpha a) \alpha^2 l^3.$$

32. By collecting all the terms together, we obtain as the total action of two particles of matter surrounded by atmospheres of repulsive caloric, estimated in the direction of the line joining their centres, and supposed of an attractive character,

$$S = -\frac{M^2}{a^2} + \frac{4\pi l^3}{3} \cdot \frac{qMP}{a^2}$$

$$+ \frac{4\pi MPA}{\alpha^2 a^2}\left\{e^{-\alpha l}(1+\alpha l) - e^{-\alpha(a-l)}(1+\alpha\,\overline{a-l})\right\}$$

$$+ \frac{2\pi AMP e^{-\alpha a}}{\alpha a^2 l^3}\left\{e^{-\alpha l}\left(a^4 + \frac{C}{\alpha} + Cl + \frac{1+\alpha a}{\alpha} l^3 + l^4\right)\right.$$

$$- e^{+\alpha l}\left(a^4 + \frac{C}{\alpha} - Cl - \frac{1+\alpha a}{\alpha} l^3 + l^4\right)$$

$$\left. - (a^3 + l^3)\left(\frac{1+\alpha a+\alpha l}{\alpha} e^{-\alpha l} - \frac{1+\alpha a - \alpha l}{\alpha} e^{\alpha l}\right)\right\}$$

$$- \frac{2\pi AMP e^{-\alpha a}}{\alpha^3 a^2 l^3}(1+\alpha a)\left\{\left(l^2 - \frac{3l}{\alpha} + \frac{3}{\alpha^2}\right) e^{\alpha l} - \left(l^2 + \frac{3l}{\alpha} + \frac{3}{\alpha^2}\right) e^{-\alpha l}\right\}$$

$$+ \frac{2\pi h A(1+\alpha a) e^{-\alpha a}}{\alpha}\{(1+\alpha l) e^{-\alpha l} - (1-\alpha l) e^{\alpha l}\}.$$

But we have seen (24) that

$$\frac{4\pi P(1+\alpha l) A e^{-\alpha l}}{\alpha^2} = M - \frac{4\pi l^3}{3} Pq;$$

hence the first line is destroyed by the first term in the second, and we get

$$S = \frac{AMP\, 2\pi e^{-\alpha a}}{\alpha a^2}\left[-\frac{2e^{\alpha l}}{\alpha}\{1 + \alpha(a-l)\}\right.$$

$$+ \frac{e^{-\alpha l}}{l^3}\left\{(1+\alpha a)\cdot\left(\frac{a^3}{\alpha} - \frac{3}{\alpha^4}\right) + \left(a^3 - 3\frac{1+\alpha a}{\alpha^3}\right)\cdot l + \frac{1+\alpha a}{\alpha}\cdot l^3 + l^4\right\}$$

$$- \frac{e^{+\alpha l}}{l^3}\left\{(1+\alpha a)\cdot\left(\frac{a^3}{\alpha} - \frac{3}{\alpha^4}\right) - \left(a^3 - 3\frac{1+\alpha a}{\alpha^3}\right)\cdot l - \frac{1+\alpha a}{\alpha}\cdot l^3 + l^4\right\}$$

$$- \frac{(a^3 + l^3)}{l^3} \left(\frac{1 + \alpha a + \alpha l}{\alpha} . e^{-\alpha l} - \frac{1 + \alpha a - \alpha l}{\alpha} . e^{\alpha l} \right)$$

$$- \frac{(1 + \alpha a)}{\alpha^2 l^3} \left\{ \left(l^2 - \frac{3l}{\alpha} + \frac{3}{\alpha^2} \right) . e^{\alpha l} - \left(l^2 + \frac{3l}{\alpha} + \frac{3}{\alpha^2} \right) . e^{-\alpha l} \right\}$$

$$+ \frac{h}{MP} (1 + \alpha a) . \left\{ (1 + \alpha l) . e^{-\alpha l} - (1 - \alpha l) . e^{\alpha l} \right\} \Big] ;$$

an expression which involves $e^{-\alpha a}$ as a factor, and which is consequently of an entirely *molecular* nature.

The above expression may be put into the following form :

$$S = \frac{AMP 2\pi e^{-\alpha a}}{\alpha a^2} [2 l e^{\alpha l}$$

$$+ (1 + \alpha a) . \{ - \frac{2 e^{\alpha l}}{\alpha} + \frac{\alpha^3 a^3 - 3}{\alpha^4 l^3} (e^{-\alpha l} - e^{\alpha l}) - \frac{3 (e^{-\alpha l} + e^{\alpha l})}{\alpha^3 l^2} + \frac{1}{\alpha} (e^{-\alpha l} + e^{\alpha l})$$

$$- \frac{a^3 + l^3}{\alpha l^3} (e^{-\alpha l} - e^{\alpha l}) + \frac{1}{\alpha^2 l^3} . \left(l^2 + \frac{3}{\alpha^2} \right) (e^{-\alpha l} - e^{+\alpha l}) + \frac{3}{\alpha^3 l^2} (e^{-\alpha l} + e^{\alpha l})$$

$$+ Qh \} + \frac{a^3}{l^2} (e^{-\alpha l} + e^{\alpha l}) + l (e^{-\alpha l} - e^{\alpha l})$$

$$- \frac{a^3 + l^3}{l^2} (e^{-\alpha l} + e^{\alpha l})]$$

$$= \frac{AMP 2\pi e^{-\alpha a}}{\alpha a^2} \left\{ (e^{-\alpha l} + e^{\alpha l}) \left(\frac{a^3 + l^3}{l^2} - \frac{a^3 + l^3}{l^2} \right) \right\}$$

$$+ \&c.$$

$$= \frac{AMP 2\pi e^{-\alpha a}}{\alpha a^2} (1 + \alpha a) . \{ (e^{-\alpha l} - e^{\alpha l}) \left(\frac{\alpha^3 a^3 - 3}{\alpha^4 l^3} + \frac{1}{\alpha} - \frac{a^3 + l^3}{\alpha l^3} \right.$$

$$\left. + \frac{\alpha^2 l^2 + 3}{\alpha^4 l^3} \right) + (e^{-\alpha l} + e^{\alpha l}) \left(\frac{3}{\alpha^3 l^2} - \frac{3}{\alpha^3 l^2} \right) + Qh \}$$

$$= \frac{AMP 2\pi e^{-\alpha a}}{\alpha^2 a^2} (1 + \alpha a) \{ (e^{-\alpha l} - e^{\alpha l}) \left(\frac{\alpha^3 a^3}{\alpha^4 l^3} - \frac{a^3}{\alpha l^3} + \frac{\alpha^2 l^2}{\alpha^4 l^3} \right) + Qh \}$$

$$= \frac{AMP 2\pi e^{-\alpha a}}{\alpha^3 a^2} (1 + \alpha a) \left\{ \frac{e^{-\alpha l} - e^{\alpha l}}{l} + Q\alpha h \right\}$$

$$= \frac{2\pi AMP\, e^{-\alpha a}}{\alpha^3 a^2}(1 + \alpha a)$$

$$\left\{\frac{ah}{MP}\left(\overline{1+\alpha l}\, e^{-\alpha l} - \overline{1-\alpha l}\, e^{\alpha l}\right) - \frac{e^{\alpha l} - e^{-\alpha l}}{l}\right\};$$

which is a very simple form, and is perfectly general, with the only exception, that we have omitted all consideration of the caloric displaced by the material particles between A and B.

33. In order to complete the expression, all that remains to be done is to find the value of k or h.

Now h evidently varies as the force on an individual particle of caloric at the surface of a material particle.

The expression parallel to x for this force is then

$$P\Sigma\left(\frac{dU}{dx} - \frac{dV}{dx}\right) = \frac{4\pi P^2 A}{\alpha^3}\Sigma\frac{X-x}{R^3}e^{-\alpha R}(1+\alpha R).$$

This may be divided into two parts, the one that which depends on the particle of matter at whose surface the force is supposed to act, the other the united effect of all the other particles. With respect to the latter, it is easy to observe that the force at the centre of the molecule is zero, and consequently *that* at the surface will be the variation of the whole force by the variation of X through the space l.

Let, therefore, F be the whole force in *one direction* on a particle of caloric in the position of the centre of the particle of matter; then will $\frac{2\,dF}{dx}\,l$ be the term in question.

34. We have therefore to find

$$\Sigma\frac{X}{R^3}e^{-\alpha R}(1+\alpha R) = G.$$

Let the whole mass be intersected by planes at the distances respectively of the particles, and parallel to yz : ϵ the distance between two consecutive particles: η the distance of the particle acted on from the first plane, $\eta + r\epsilon$ is the distance of any plane.

Let the plane be divided into annuli of which the radius drawn from the line of intersection with the axis of x is ρ, then

$$\frac{2\pi\rho\delta\rho}{\epsilon^2}\frac{\eta + r\epsilon}{R^3}e^{-aR}(1 + aR)$$

is the part of G for this annulus;

$$\therefore \text{ part for plane } = \frac{2\pi}{\epsilon^2}\int_0^\infty \rho d\rho (\eta + r\epsilon) e^{-aR}\frac{(1 + aR)}{R^3}$$

$$= \frac{2\pi}{\epsilon^2}\int_{\eta+r\epsilon}^\infty dR (\eta + r\epsilon) e^{-aR}\frac{(1 + aR)}{R^2}$$

$$= \frac{2\pi}{\epsilon^2}\frac{(\eta + r\epsilon) e^{-a(\eta+r\epsilon)}}{\eta + r\epsilon}$$

$$= \frac{2\pi}{\epsilon^2} e^{-a(\eta+r\epsilon)}.$$

$$\text{Hence, } G = \frac{2\pi}{\epsilon^2} e^{-a\eta}\Sigma_0^\infty e^{-ar\epsilon}$$

$$= \frac{2\pi}{\epsilon^2}\cdot\frac{e^{-a\eta}}{1 - e^{-a\epsilon}},$$

$$\text{and } F = \frac{H}{\epsilon^2(1 - e^{-a\epsilon})} e^{-a\eta};$$

$$\therefore \frac{dF}{d\eta} = -\frac{ale^{-a\eta}}{\epsilon^2(1 - e^{-a\epsilon})},$$

so that the term in question is

$$\frac{2alH}{\epsilon^2(1 - e^{-a\epsilon})} e^{-a\epsilon}.$$

The other part of h is $\dfrac{4\pi PA}{a^2} e^{-al}\dfrac{1 + al}{l^2}$

$$= \frac{1}{l^2}(M - \frac{4\pi l^3}{3} Pq).$$

If, however, it were required to find the value of h for a particle not far from the surface of the medium, it would be necessary that we should know the law of variation of density at the surface. For the sake of obtaining the former, let the density be supposed uniform, then instead of $\frac{Le^{-\alpha\epsilon}}{\epsilon^2(1-e^{-\alpha\epsilon})}$ the term will become

$$\frac{Le^{-\alpha\epsilon}}{\epsilon^2(1-e^{-\alpha\epsilon})} - \frac{R}{\epsilon^2}\Sigma_p^\infty e^{-\alpha n\epsilon} = \frac{L}{\epsilon^2(1-e^{-\alpha\epsilon})}(1 - re^{-\alpha p\epsilon}),$$

the particle being distant by $p\epsilon$ from the surface:

by substituting this expression, the force of attraction becomes

$$S = \frac{4\pi AMPe^{-aa}(1+aa)}{a^3a^2}\left[\left\{K - L\frac{1-re^{-\alpha p\epsilon}}{\epsilon^2(1-e^{-\alpha\epsilon})}\right\}\left(1 - \frac{4\pi l^3P}{3M}q\right) - \frac{(e^{-al}-e^{al})}{l}\right].$$

This is a very simple expression for the mutual action of two particles of matter. As ϵ diminishes, the attractive part of the force diminishes, so that there is a resistance to the approach of the particles towards each other.

Suppose a particle situated at or near the confines of the medium to be in equilibrium: then the sum of all expressions similar to the above, taken throughout the medium must equal zero.

35. I shall only very approximately find the action on a particle bounding a medium: for it is obvious that in general the force on it from the surrounding caloric will differ widely from the force on a particle in the interior of the medium; the former depending only on the particles on one side of that in question, the latter depending on two sets of particles acting in opposite directions, and tending to counteract each other's efforts. On this account there will in general be a rapid diminution of density towards the surface of the medium. The law of this diminution I have attempted to investigate, but from the circumstance that the resulting equations involve the mixed differences of discontinuous functions, I have not hitherto arrived at any satisfactory conclusion. I shall therefore satisfy myself with finding the force on a particle bounding a medium, on the supposition that the medium is homogeneous.

The expression for the attraction of any particle, is of the form

$$e^{-\alpha a} \frac{Ha - m}{a^2}.$$

Now whatever be the form of the bounding surface, it is obvious that unless the sphere of sensible action be great, it will suffice to consider it plane and extending to infinity: we shall then have to estimate the aggregate force on a particle resolved perpendicular to the bounding plane.

Let the atom under consideration be the centre of a spherical surface to radius a: take an annulus of this surface such that the radius vector drawn to it makes the angle θ with the bounding plane:

$$\text{the area of this annulus} = 2\pi a^2 \cos\theta \, d\theta,$$

$$\text{and the number of particles in it} = \frac{2\pi a^2}{\epsilon^2} \cos\theta \, d\theta;$$

hence the attraction on the particle in question resolved perpendicularly to the bounding plane,

$$= \frac{2\pi a^2 \sin\theta \cos\theta \, d\theta}{\epsilon^2} \frac{e^{-\alpha a}}{a^2} (Ha - m),$$

and the whole force due to the particles in the hemispherical surface, is

$$\frac{\pi Ha}{\epsilon^2} e^{-\alpha a} - \frac{\pi}{\epsilon^2} . me^{-\alpha a}.$$

In order to find the whole attraction on the given particle, we must find the sum of all similar expressions taken through the whole mass: which is

$$\frac{\pi H}{\epsilon^2} \{\epsilon e^{-\alpha\epsilon} + 2\epsilon e^{-2\alpha\epsilon} + \ldots\} - \frac{\pi m}{\epsilon^2} e^{-\alpha\epsilon}(1 + e^{-\alpha\epsilon} + \ldots)$$

$$= \frac{\pi H}{\epsilon^2} \epsilon e^{-\alpha\epsilon}(1 + 2e^{-\alpha\epsilon} + 3e^{-2\alpha\epsilon} + \ldots) - \frac{\pi m e^{-\alpha\epsilon}}{\epsilon^2 (1 - e^{-\alpha\epsilon})}$$

$$= \frac{\pi H}{\epsilon} e^{-\alpha\epsilon} \left(\frac{1}{1 - e^{-\alpha\epsilon}}\right)^2 - \frac{\pi m}{\epsilon^2} \frac{e^{-\alpha\epsilon}}{1 - e^{-\alpha\epsilon}}.$$

This is the expression which ought to remain constant, whatever be ϵ, so long as the temperature is so. It is obvious that it varies directly as H, which involves $m' - n'q$.

Moreover the density of the caloric at any point

$$= q - A\Sigma\frac{e^{-aR}}{R}$$

$$= q - \frac{\pi A}{\epsilon}\frac{e^{-a\epsilon}}{(1-e^{-a\epsilon})^2}$$

$$= q - \frac{\pi}{\epsilon}(A_{,} - B_{,}q)\frac{e^{-a\epsilon}}{(1-e^{-a\epsilon})^2}.$$

We see, then, that the density of caloric is not a proper measure of the temperature, although if ϵ be small, the variation of density will be a proper measure of that of the temperature; each depending on $\frac{e^{-a\epsilon}}{\epsilon^3}$, or on the density of the material particles, which result I obtained in a popular manner in my Theory of Heat, p. 166, and found some remarkable consequences to accrue from it.

36. From the expression for the force, it follows directly that if q increase, which it must do if the temperature be increased in whatever way that increase be measured, the attractive force diminishes, but this diminution will also be accompanied by a diminution of the repulsive force provided a increase, and that too, not by the variation of the common factor $e^{-aa}\frac{(1+aa)}{a^2}$, but by the diminution of $\frac{1}{\epsilon^2(1-e^{-a\epsilon})}$. Hence, we perceive that the same series of particles will by an increase in their mutual distances, exert actions on any one particle just sufficient to retain it in equilibrium, notwithstanding that the quantity of caloric has been increased. This fully explains the necessity of expansion by heat.

37. It was my original intention in drawing up the present Memoir, to have extended the investigations to a set of combinations of particles, such as I have supposed to unite in the formation of crystals in my Theory of Heat, p. 174, but the subject is so extensive, that I am at present obliged to postpone it for want of time. I will only make one remark on the subject, which is this, that if in a binary combination the lines joining the centres of each pair of particles are parallel to one another, it is obvious, that the attractive force on a

particle estimated in the direction of this line, will be different from the force in a plane perpendicular to it, not only in value, but also in form.

For the supposition of contact between the material particles amounts to that of exclusion of caloric particles, and consequently we cannot estimate the action of each particle on every other, as though these two were the only ones of the system, but must add or subtract from that action a force due to the caloric displaced at the point of junction; and further, the repulsion of the caloric surrounding a particle, must be diminished in the direction of the line joining the centres, on account of the quantity displaced by the neighbouring particle. The two sets of forces will therefore be totally different in form in the two directions. In that joining the centres of the particles, the variation of the attraction for a variation of q as well as for a variation of a, will be very considerable, whereas in a perpendicular direction both variations will be small. But besides this, it will be seen that the term which in Art. 29. completely destroyed $\dfrac{M^2 - \frac{4\pi l^3}{3} qMP}{a^2}$, being a term arising from the displaced caloric, will not now be sufficient to destroy it on account of the accompanying particle, consequently a very small attraction varying inversely as the square of the distance will remain. This attraction cannot have a sensible value as compared with the other terms when the distance is small; but when the distance is finite, the rapid diminution of $e^{-\alpha a}$ renders the other terms very much smaller than this, and at a considerable distance this term is the only one sensible: at such distances then, the force varies inversely as the square of the distance. Thus all the known laws, as well of attraction as of cohesion, are explained by the Newtonian hypothesis.

QUEENS' COLLEGE,
March 10, 1838.

NOTE (a). Let A be the centre of the attracting, B of the attracted particle $AB = a$, $AQ = \rho$, $AC = R$ the radius of any sphere, $V =$ the volume of $B = \frac{4\pi l^3}{3}$ – angle $QAM = \phi$.

III. *On Rolling Curves.* *By* HAMNETT HOLDITCH, M.A., *Fellow of Caius College, and of the Cambridge Philosophical Society.*

[Read *December* 10, 1838.]

IN the fifth volume of the *Acta Petropolitana*, Euler referred to a class of curves which, when caused to turn round fixed centres, possessed the property of communicating motion to each other without friction; he deduced also their characteristic property, that the point of contact remains always in the straight line joining their centres: he has not however followed out the investigation so as to furnish actual forms of curves, neither has this been done by any other writer that I am aware of, and consequently no method exists by which such curves can be obtained. But as they are practically employed in a manner which I shall proceed to explain, and commonly found by a tentative process, it appeared worth while to search for forms and rules for their construction, independently of the analytical interest that may be supposed to attach to such investigation.

Let Anm, $Bn_{,}m_{,}$ (Fig. 1.) be two curves capable of rolling together, and having their centres of rotation A and B fixed at a distance equal to the sum of their apsidal distances, Am being a long and $Bn_{,}$ a short apsidal distance, then if nAm be caused to turn round in the direction of the arrow, it will press against $Bn_{,}m_{,}$ and communicate a rotation to it. This action will, however, cease when the point m has reached $n_{,}$; for beyond that point the radii of mAn will diminish, and its circumference begin to recede from the other curve.

No continuous motion of B can therefore be derived from that of A, if they be *continuous* curves, unless their outlines be treated like the pitch lines of ordinary wheels, and be indented with small teeth at regular distances; these teeth, as in the usual forms, projecting nearly as far beyond the pitched line or circumference as they extend within it. If this be done, it will be found that the circumference of A will retain its hold on that of B in all positions, as well on the receding as on the advancing sides of the curve. A continuous uniform rotation of one curve will produce a rotation of the other, not uniform, but continually varying in its angular velocity, as the ratio of the radius of A to that of B; this becomes then a commodious contrivance for converting an equable angular velocity into an unequal one, and is sometimes so used by Mechanists. Fergusson's well-known Cometarium was constructed on this principle: it is to be found in use in some silk machinery, where it is introduced for the purpose of correcting the unequal action of the common excentric in laying the silk upon the bobbins; it has also been used by Messrs Bacon and Donkin, in their printing machinery. I am informed by Professor Willis, who drew my attention to the subject of these curves, and furnished me with the above practical information, that the copious collections of Messrs Lanz and Betancourt, and that of Borgnis, furnish no example of the application of rolling curves to the purposes of machinery; which may therefore be considered to have been unknown to them.

When two such curves roll on each other, let r be the distance of their point of contact from the centre of rotation of the first curve, and θ the angle made by r with a fixed radius; then $\frac{rd\theta}{dr}$ is the tangent of the angle the curve makes with r; and $r_{,}$ and $\theta_{,}$ being corresponding quantities in the second curve, $\frac{r_{,}d\theta_{,}}{dr_{,}}$ is the tangent of the angle it makes with $r_{,}$, and as r and $r_{,}$ are in the same straight line, and the curves must have a common tangent at the point of contact, these two angles must be equal, and

$$\therefore \frac{rd\theta}{dr} = \frac{r_{,}d\theta_{,}}{dr_{,}};$$

Also, if c be the distance of the centres, $r + r_{,} = c$, and

$$\therefore\ dr^2 + r^2 d\theta^2 = dr^2\left(1 + \frac{r^2 d\theta^2}{dr^2}\right) = dr_{,}^2\left(1 + \frac{r_{,}^2 d\theta_{,}^2}{dr_{,}^2}\right) = dr_{,}^2 + r_{,}^2 d\theta_{,}^2,$$

or the differentials of the lengths of those parts of the curves which have been in contact are equal, and

$$\therefore\ r + r_{,} = c, \text{ and } \frac{r d\theta}{dr} = \frac{r_{,} d\theta_{,}}{dr_{,}},$$

are equations which contain the analytical conditions of such curves.

We will first consider the case of two equal and similar curves rolling on each other. Since $\frac{d\theta}{dr}$ is some function of r, $\frac{r d\theta}{dr}$ must also be a function of r, let it $= f(r)$; and as $r_{,}$ and $\theta_{,}$ belong to a point in a similar and equal curve,

$$\therefore\ \frac{r_{,} d\theta_{,}}{dr_{,}} = f(r_{,});\ \text{ and } r_{,} = c - r;\quad \therefore\ f(r) = f(c - r),$$

the solution of which equation is $f(r) = \phi(r, c - r)$; any symmetric function of r and $c - r$, and if any form be given to ϕ in the equation $\frac{r d\theta}{dr} = \phi(r, c - r)$, the integration of the latter will give the equation to a curve having the proposed property. If we suppose it to have greater and less apsidal distances a and b, which most curves which can practically be used, must possess; then, as in revolving the greater apsidal distance of one must come into contact with the less apsidal distance of the other, $a + b = c$;

Now $(a - r).(r - b) = (a + b).r - r^2 - ab = r.(c - r) - ab$, is a symmetric function of r and $c - r$; and as at apses $\frac{dr}{d\theta} = 0$, if we assume $\frac{r d\theta}{dr} = \frac{X(r, c - r)}{\sqrt{(a - r).(r - b)}}$ where $X(r, c - r)$ is any symmetric function of r and $c - r$ which is not divisible by $\sqrt{(a - r).(r - b)}$, the curve will be confined between the apsidal distances; and supposing also, that X contains only positive integral powers of r and $c - r$, this last equation can always be integrated in finite terms.

If $r - \frac{c}{2} = x$, any component part of X as

$$r^m.(c-r)^n + r^n.(c-r)^m = \left(\frac{c}{2}+x\right)^m.\left(\frac{c}{2}-x\right)^n + \left(\frac{c}{2}+x\right)^n.\left(\frac{c}{2}-x\right)^m$$

$$= \left(\frac{c}{2}+x\right)^{m-n}.\left(\frac{c}{2}-x\right)^{m-n}.\left\{\left(\frac{c}{2}+x\right)^n + \left(\frac{c}{2}-x\right)^n\right\}$$

$$= \left(\frac{c^2}{4}-x^2\right)^{m-n}.2\left\{\left(\frac{c}{2}\right)^n + n.\frac{n-1}{2}.\left(\frac{c}{2}\right)^{n-2}.x^2 + \ldots\ldots\right\},$$

consists of even powers of x only, and therefore X will contain no negative powers of x, and will be of the form

$$X(r, c-r) = k_{,} + k\left(r - \frac{a+b}{2}\right)^2 + \ldots\ldots$$

and limiting the investigation, for the sake of simplicity, to the first two terms,

$$\text{we have } d\theta = \frac{k_{,} + k.\left(r - \frac{a+b}{2}\right)^2}{r\sqrt{(a-r).(r-b)}}.dr.$$

To integrate this, let $\frac{a+b}{2} = \alpha$, $\frac{a-b}{2} = \beta$;

$$\therefore (a-r).(r-b) = (a+b).r - r^2 - ab = 2\alpha r - r^2 - \alpha^2 + \beta^2 = \beta^2 - (r-\alpha)^2;$$

$$\therefore d\theta = \frac{k_{,} + k.(r-\alpha)^2}{r\sqrt{\beta^2 - (r-\alpha)^2}}.dr.$$

Assume $r - \alpha = \beta\cos\phi$, then $d\theta = -\frac{k_{,} + k\beta^2\cos^2\phi}{\alpha + \beta\cos\phi}.d\phi$

$$= -\frac{k_{,} + k.(\alpha + \beta\cos\phi - \alpha)^2}{\alpha + \beta\cos\phi}.d\phi$$

$$= -\frac{k_{,} + k\alpha^2}{\alpha + \beta\cos\phi}.d\phi + k\alpha d\phi - k\beta\cos\phi d\phi.$$

Now $\int \frac{d\phi}{\alpha + \beta \cos\phi}$ (See Professor Peacock's Examples)

$$= \frac{2}{\sqrt{\alpha^2 - \beta^2}} . \tan^{-1} \frac{(\alpha - \beta) . \tan \frac{\phi}{2}}{\sqrt{\alpha^2 - \beta^2}}$$

$$= \frac{2}{\sqrt{ab}} . \tan^{-1} \left(\sqrt{\frac{b}{a}} . \tan \frac{\phi}{2}\right),$$

and $\tan \frac{\phi}{2}$ from the equation

$$r - \frac{a+b}{2} = \frac{a-b}{2} . \cos\phi \text{ is found to be } = \sqrt{\frac{a-r}{r-b}};$$

$$\therefore \theta = - \frac{2k_, + k . \frac{(a+b)^2}{2}}{\sqrt{ab}} . \tan^{-1} \sqrt{\frac{b}{a}} . \sqrt{\frac{a-r}{r-b}} + \alpha k\phi - k\beta \sin\phi + C$$

$$= - \frac{2k_, + k . \frac{(a+b)^2}{2}}{\sqrt{ab}} . \tan^{-1} \sqrt{\frac{b}{a}} . \sqrt{\frac{a-r}{r-b}}$$

$$- k\sqrt{(a-r).(r-b)} + k.(a+b).\tan^{-1}\sqrt{\frac{a-r}{r-b}} + C,$$

$$\text{or } \theta = \frac{2k_, + k . \frac{(a+b)^2}{2}}{\sqrt{ab}} . \tan^{-1} \sqrt{\frac{a}{b}} . \sqrt{\frac{r-b}{a-r}}$$

$$- k . \sqrt{(a-r).(r-b)} - k.(a+b).\tan^{-1}\sqrt{\frac{r-b}{a-r}}, \qquad (1),$$

where θ is measured from the smaller apse, is the equation to a class of curves, which for the present may be called *self-rolling curves*.

If $k_, = \sqrt{ab}$, and $k = 0$, $\theta = 2 . \tan^{-1} \sqrt{\frac{a}{b}} . \sqrt{\frac{r-b}{a-r}}$;

and $r = \dfrac{ab}{a\cos^2\dfrac{\theta}{2} + b\sin^2\dfrac{\theta}{2}} = \dfrac{2ab}{(a+b)+(a-b).\cos\theta}$ the equation to an ellipse round the focus, which is known to be capable of rolling upon another equal and similar ellipse.

$$\text{Hence } \theta = \frac{2k_{,}}{\sqrt{ab}}.\tan^{-1}\sqrt{\frac{a}{b}}.\sqrt{\frac{r-b}{a-r}}$$

is the equation to the curve constructed in the ninth section of Newton's *Principia*, which is therefore a *self-rolling curve*.

In the equation found above, if $r = b$,

$$\theta = \left\{\frac{2k_{,}}{\sqrt{ab}} + \frac{k.(a+b)^2}{2\sqrt{ab}} - k.(a+b)\right\}.m\pi,$$

so that the minor apsidal distances recur, the angular distances between them being $= \left\{\dfrac{2k_{,}}{\sqrt{ab}} + \dfrac{k.(a+b)^2}{2\sqrt{ab}} - k.(a+b)\right\}.\pi.$

If $r = a$,

$$\theta = \left\{\frac{2k_{,}}{\sqrt{ab}} + \frac{k.(a+b)^2}{2\sqrt{ab}} - k.(a+b)\right\}.(2m+1).\frac{\pi}{2},$$

and the major apsidal distances recur and bisect the angles between the minor distances: and if that portion of the curve between two minor distances, including as they do, a major distance between them, be called a *Lobe*, the number of lobes in a revolution

$$= 2\pi \div \left\{\frac{2k_{,}}{\sqrt{ab}} + \frac{k.(a+b)^2}{2\sqrt{ab}} - k.(a+b)\right\}.\pi,$$

and in order that the curve may return into itself and so be capable of successive revolutions, this must be an integer $= n$;

$$\text{and } \therefore \frac{2k_{\prime}}{\sqrt{ab}} + \frac{k.(a+b)^2}{2\sqrt{ab}} - k.(a+b) = \frac{2}{n}, \qquad (2),$$

and the equation to a self-rolling curve of n lobes is

$$\theta = \left\{\frac{2}{n} + k.(a+b)\right\}.\tan^{-1}\sqrt{\frac{a}{b}}.\sqrt{\frac{r-b}{a-r}}$$

$$-k\sqrt{(a-r).(r-b)} - k.(a+b).\tan^{-1}\sqrt{\frac{r-b}{a-r}}, \qquad (3),$$

where the constants a, b, n, k may be assumed at pleasure, and for any value of r, the corresponding value of θ will be obtained, and the curve may then be traced by points.

The different forms the curves may assume will be best illustrated by an example; thus if $a = 10$, and $b = 1$, and $k = 11k_2$, then if

$$r = 1, \quad \theta = 0,$$

$$r = 2, \quad \theta = \frac{96^0.4}{n} + 14^0 k_2,$$

$$r = 3, \quad \theta = \frac{118.8}{n} + 11.9k_2,$$

$$r = 4, \quad \theta = \frac{131.8}{n} + 8.7k_2,$$

$$r = 5, \quad \theta = \frac{141}{n} + 5.4k_2,$$

$$r = 6, \quad \theta = \frac{148.4}{n} + 2.7k_2,$$

$$r = 7, \quad \theta = \frac{154.8}{n} + .6k_2,$$

$$r = 8, \quad \theta = \frac{160.8}{n} - .9k_2,$$

$$r = 9, \quad \theta = \frac{167.2}{n} - 1.6k_2,$$

$$r = 10, \quad \theta = \frac{180}{n}.$$

If $n = 1$ the curve is of one lobe, and if also $k_2 = 0$ it is an ellipse; and examples are given when $k_2 = 2, 4, 10$ and 20 in figures 2, 3, 4, 5, C being the fixed centre; if $k_2 = -2, -4, -6, -15, -20$, the representations are given in figures 6, 7, 8, 9, 10, in all which figures, only the upper half of the lobe is drawn, as the lower is similar and equal to it: and although in some of the figures the radius vector has swept over more than half a circumference, it has returned so that the semi-lobe has terminated when $\theta = \pi$.

If $n = 2, 3, 4$, &c., curves of 2, 3, 4, &c., lobes may be traced from the above table, and are readily laid down.

If $a = 16.95$, and $b = 6.95$, the following is another table for a great variety of self-rolling curves:

When $r - b = 1$, $\theta = \dfrac{55^\circ}{n} + 45^\circ.6k$,

2, $\theta = \dfrac{76}{n} + 43.3k$,

3, $\theta = \dfrac{91.4}{n} + 36.2k$,

4, $\theta = \dfrac{103.8}{n} + 22.9k$,

5, $\theta = \dfrac{114.8}{n} + 10k$,

6, $\theta = \dfrac{124.8}{n} - 3.4k$,

7, $\theta = \dfrac{134.6}{n} - 11.6k$,

8, $\theta = \dfrac{144.4}{n} - 18.9k$,

9, $\theta = \dfrac{156.2}{n} - 16.5k$;

and an hour is sufficient to make a table for any assumed apsidal distances. It will be seen that if k be positive, as k increases, the curves bulge at the greater apse; if k be negative and increases, the curves bulge more and more at the lower apse; this will afterwards appear from the consideration of the radius of curvature.

Fig. (25) is an example of a two-lobed curve.

In some cases, as in figures 8, 9, 10, the semilobe commences at the minor apse by a retrograde motion of the radius vector, and terminates in such cases by a retrograde motion at the major apse: for let A be the value of θ near the smaller apse when $r = b + z$, and B the value of $\pi - \theta$ near the major apse when $r = a - z$, z being a very small quantity, then we get from the equation to the curve;

$$A = \sqrt{\frac{a}{b}} \cdot \sqrt{\frac{z}{a-b}} \cdot \left(\frac{2}{n} + k(\sqrt{a} - \sqrt{b})^2\right)$$

$$B = \sqrt{\frac{b}{a}} \cdot \sqrt{\frac{z}{a-b}} \cdot \left(\frac{2}{n} + k(\sqrt{a} - \sqrt{b})^2\right), \qquad (4);$$

and therefore if k be positive, A and B are positive; and if k be negative, A and B will be both positive, or both negative; for $\frac{A}{B} = \frac{a}{b}$, so that if a portion of the upper semilobe is below the axis at one apse, there will also be a portion below at the other apse.

As k increases the curves run into hooks, the points of which have a tangent passing through the centre, and there can only be two in each semilobe determined from the equation $k_{,} + k \cdot \left(r - \frac{a+b}{2}\right)^2 = 0$, for at these *tangent points* $\frac{d\theta}{dr} = 0$, and this equation has only two roots, the sum of which is $a + b$ and therefore *in rolling they come into contact.*

If the value of $k_{,}$ from equation (2) be substituted in this last, the distances of the tangential points from the centre are

$$r = \frac{a+b}{2} \pm \sqrt{-\frac{\sqrt{ab}}{nk} + \frac{(a+b)}{4} \cdot (\sqrt{a} - \sqrt{b})^2}. \qquad (5).$$

If k be positive, there are no tangential points unless k is equal to, or greater than,

$$\frac{4\sqrt{ab}}{n(a+b).(\sqrt{a}-\sqrt{b})^2};$$

they begin at $r=\frac{a+b}{2}$, and as k increases, one moves nearer to, and the other farther from the centre; and when k is infinite,

$$r=\frac{a+b}{2}\pm\frac{\sqrt{a+b}}{2}(\sqrt{a}-\sqrt{b}).$$

If k be negative, the values of r in equation (5) must be within the limits of the curve, and therefore there are no tangent points unless $k>\frac{2}{n(\sqrt{a}-\sqrt{b})^2}$, and if k be infinite their distances are the same as when k is positive: comparing this condition with equation (4) it will be seen that when k is negative and the curve is retrograde at the apses, there are always tangent points.

Other forms of self-rolling curves may be found, as

$$r=\frac{a}{1+k\theta^2},\quad \theta=ar^2-br,$$

and $\theta = A.\text{hyp}\log r + Bar + (Ca^2-B).\frac{r^2}{2}-\frac{2aC}{3}.r^3+\frac{Cr^4}{4}$,

the latter including the logarithmic spiral.

Fig. 22 is a self-rolling curve, where the minor apsidal distance vanishes, and rolls round the point C in its circumference.

We will now proceed to the consideration of rolling curves when they are not necessarily similar and equal to each other. If c be the distance of their centres, and $\frac{rd\theta}{dr}=f(r)$ be the differential equation of one of the curves, and $r_{,}$ and $\theta_{,}$ belong to a curve that will roll with the former, then, since

$$\frac{rd\theta}{dr}=\frac{r_{,}d\theta_{,}}{dr},\text{ and } f(r)=f(c-r),$$

it follows from what has been observed before, that

$$\frac{r_{,} d\theta_{,}}{dr_{,}} = f(c - r_{,}),$$

will be the differential equation of the latter; and any form being given to f, the integration of these equations will be the equations to a pair of rolling curves; and for other values of c, other curves may be found, and so a system formed.

The equation to one of the curves being assumed to be that which has been found for self-rolling curves, viz.

$$\frac{rd\theta}{dr} = \frac{k_{,} + k\left(r - \frac{a+b}{2}\right)^2}{\sqrt{(a-r).(r-b)}};$$

the equation to the other will therefore be

$$\frac{r_{,}d\theta_{,}}{dr_{,}} = \frac{k_{,} + k.\left(c - r_{,} - \frac{a+b}{2}\right)^2}{\sqrt{(a-c+r_{,}).(c-r_{,}-b)}};$$

$$\text{let } c - b = a_{,}, \text{ and } c - a = b_{,}; \quad \therefore a_{,} - b_{,} = a - b,$$

$$\text{and } c - \frac{a+b}{2} = \frac{a_{,}+b_{,}}{2};$$

$$\therefore \frac{r_{,}d\theta_{,}}{dr_{,}} = \frac{k_{,} + k.\left(\frac{a_{,}+b_{,}}{2} - r_{,}\right)^2}{\sqrt{(a_{,}-r_{,}).(r_{,}-b_{,})}} = \frac{k_{,} + k.\left(r_{,} - \frac{a_{,}+b_{,}}{2}\right)^2}{\sqrt{(a_{,}-r_{,}).(r_{,}-b_{,})}},$$

which is of the same form as the differential equation of the assumed curve, and therefore if $n_{,}$ be the number of its lobes,

$$\theta_{,} = \left\{\frac{2}{n_{,}} + k.(a_{,}+b_{,})\right\}.\tan^{-1}\sqrt{\frac{a_{,}}{b_{,}}}.\sqrt{\frac{r_{,}-b_{,}}{a_{,}-r_{,}}}$$

$$- k.\sqrt{(a_{,}-r_{,}).(r_{,}-b_{,})} - k.(a_{,}+b_{,}).\tan^{-1}.\sqrt{\frac{r_{,}-b_{,}}{a_{,}-r_{,}}},$$

is the equation to a curve which will roll with the former, the equations of condition being $a_{\prime} - b_{\prime} = a - b$,

$$\text{and} \quad \frac{2k_{\prime}}{\sqrt{a_{\prime}b_{\prime}}} + \frac{k}{2}\cdot\frac{(a_{\prime}+b_{\prime})^2}{\sqrt{a_{\prime}b_{\prime}}} - k\,.\,(a_{\prime}+b_{\prime}) = \frac{2}{n_{\prime}}.$$

Hence, if k, $k_{\prime}$ and $a - b$ be any assumed constant quantities, the values of a and b may be found for $n = 1, 2, 3$, &c. from the equation

$$\frac{2k_{\prime}}{\sqrt{ab}} + \frac{k}{2}\cdot\frac{(a+b)^2}{\sqrt{ab}} - k\,.\,(a+b) = \frac{2}{n},$$

by the solution of a cubic equation, as will be easily seen, and the curves constructed from the equation

$$\theta = \left\{\frac{2}{n} + k\,.\,(a+b)\right\}.\tan^{-1}\sqrt{\frac{a}{b}}\,.\sqrt{\frac{r-b}{a-r}}$$

$$- k\sqrt{(a-r)\,.\,(r-b)} - k\,.\,(a+b)\,.\tan^{-1}\sqrt{\frac{r-b}{a-r}},$$

and a system of wheels or curves thus found will roll together in pairs or in any combinations.

When there are tangential points in one wheel, there will be corresponding ones in all of the same system, and in rolling they will come into contact with each other; for those of one wheel are found by making $f(r) = 0$, and if a be a root of this equation, $c - a$, or $a_{\prime}$ will be a root of the equation $f(c - r) = 0$, or of $f(r_{\prime}) = 0$, and $\therefore\ a + a_{\prime} = c$.

Forms of wheels are readily found from assumed values of k and $k_{\prime}$: or if the dimensions of a pair of wheels be assumed, k and $k_{\prime}$ may be found from equation (2);

Thus, if $\left.\begin{matrix} n = 1, & b = 1 \\ n_{\prime} = 3, & b_{\prime} = 5 \end{matrix}\right\} a = 10$, and $\therefore\ a_{\prime} = 14\}$. Fig. (11).

If $\left.\begin{matrix} n = 1, & b = 1 \\ n_{\prime} = 3, & b_{\prime} = 4 \end{matrix}\right\} a = 10$, and $\therefore\ a_{\prime} = 13\}$. Fig. (12).

If $\left.\begin{matrix} n = 1, & b = 1 \\ n_{\prime} = 1, & b_{\prime} = 4 \end{matrix}\right\} a = 10$, and $\therefore\ a_{\prime} = 13\}$. Fig. (13);

in all which cases each curve is also a self-rolling curve.

In this latter example, it will be observed, that the curves are retrograde at the apses, which will be the case with all *unequal* curves that are made to roll together, if they have the *same* number of lobes;

$$\text{for } k_{,} + k.\left(\frac{a-b}{2}\right)^2 = \frac{\dfrac{\sqrt{ab}}{n}.\sqrt{a_{,}b_{,}}.(\sqrt{a_{,}}-\sqrt{b_{,}})^2 - \dfrac{\sqrt{a_{,}b_{,}}}{n_{,}}.\sqrt{ab}.(\sqrt{a}-\sqrt{b})^2}{\sqrt{a_{,}b_{,}}.(\sqrt{a_{,}}-\sqrt{b_{,}})^2 - \sqrt{ab}.(\sqrt{a}-\sqrt{b})^2},$$

from equations (2), and if $n = n_{,}$, this expression may be proved to be negative when a and $a_{,}$, and therefore b and $b_{,}$, are unequal: and since

$$\frac{2}{n} + k(\sqrt{a}-\sqrt{b})^2 = \frac{2}{\sqrt{ab}}.\left\{k_{,} + k.\left(\frac{a-b}{2}\right)^2\right\},$$

$$\text{and } \frac{2}{n_{,}} + k(\sqrt{a_{,}}-\sqrt{b_{,}})^2 = \frac{2}{\sqrt{a_{,}b_{,}}}.\left\{k_{,} + k.\left(\frac{a-b}{2}\right)^2\right\};$$

therefore, by equations (4), the curves are retrograde at the apses.

If two curves roll one *within* the other round fixed centres whose distance is c, then

$$r = r_{,} \pm c, \quad \text{and} \quad \frac{rd\theta}{dr} = \frac{r_{,}d\theta_{,}}{dr_{,}}, \quad \text{and if} \quad \frac{rd\theta}{dr} = \frac{k_{,} + k.\left(r - \dfrac{a+b}{2}\right)^2}{\sqrt{(a-r).(r-b)}},$$

be taken for the differential equation of one;

$$\frac{r_{,}d\theta_{,}}{dr_{,}} = \frac{k_{,} + k.\left(r_{,} \pm c - \dfrac{a+b}{2}\right)^2}{\sqrt{(a - r_{,} \mp c).(r_{,} \pm c - b)}},$$

will be that of the other;

$$\text{let } a \mp c = a_{,}; \quad \text{and } b \mp c = b_{,};$$

$$\therefore\ a - b = a_{,} - b_{,}, \quad \text{and} \quad \frac{a+b}{2} \mp c = \frac{a_{,}+b_{,}}{2};$$

$$\text{and } \frac{r_{,}d\theta_{,}}{dr_{,}} = \frac{k_{,} + k.\left(\dfrac{a_{,}+b_{,}}{2} - r_{,}\right)^2}{\sqrt{(a_{,}-r_{,}).(r_{,}-b_{,})}} = \frac{k_{,} + k.\left(r_{,} - \dfrac{a_{,}+b_{,}}{2}\right)^2}{\sqrt{(a_{,}-r_{,}).(r_{,}-b_{,})}},$$

and therefore, the equations will be the same as for those that roll externally, and the equations of condition are also the same, and consequently *all curves* (whose equations are of the forms that have been considered) *that roll externally, are also capable of rolling on each other internally;* in the latter case, the major axes come into contact;

$$\text{for if } r_{,} = a_{,}, \quad r = r_{,} \pm c = a_{,} \pm c = a_{,} + \frac{a+b}{2} - \frac{a_{,}+b_{,}}{2}$$

$$= \frac{a_{,}-b_{,}}{2} + \frac{a+b}{2} = \frac{a-b}{2} + \frac{a+b}{2} = a: \text{ and if } r_{,} = b_{,}, \quad r = b.$$

The curves in fig. (13) will therefore roll in the positions, figs. (14) and (15), which are two different attitudes of the curves.

It will be necessary hereafter to know the radii of curvature at different points of the curve;

$$\text{Let } \frac{rd\theta}{dr} = \frac{k_{,} + k \cdot \left(r - \frac{a+b}{2}\right)^2}{\sqrt{(a-r)\cdot(r-b)}} = \frac{X}{\sqrt{Y}} = \frac{s}{\sqrt{1-s^2}},$$

where s is the sine of the angle between the curve and radius vector, then $p = sr$, p being the perpendicular upon the tangent;

$$\therefore \frac{dp}{rdr} = \frac{s}{r} + \frac{ds}{dr},$$

$$\frac{1}{s^2} = 1 + \frac{Y}{X^2};$$

therefore, to find the radius of curvature at the tangent points where $s = 0$, and $X = 0$,

$$\frac{ds}{s^3 dr} = \frac{Y}{X^3} \cdot \frac{dX}{dr},$$

$$\frac{ds}{dr} = \frac{s^3}{X^3} \cdot \frac{YdX}{dr},$$

$$\frac{X^2}{s^2} = X^2 + Y = Y; \quad \therefore \frac{ds}{dr} = \frac{dX}{\sqrt{Y}.dr},$$

and $\dfrac{dX}{dr} = 2k.\left(r - \dfrac{a+b}{2}\right) = 2k.\sqrt{\dfrac{-k_{,}}{k}}$ when $X = 0$;

also $Y = (a-r).(r-b) = \dfrac{l^2}{4} - \left(r - \dfrac{a+b}{2}\right)^2 = \dfrac{l^2}{4} + \dfrac{k_{,}}{k}$ where $l = a - b$;

therefore, if R = radius of curvature at the tangent points,

$$\frac{1}{R} = \frac{dp}{rdr} = \frac{ds}{dr} = \frac{2k\sqrt{-k_{,}}}{\sqrt{k_{,} + k\dfrac{l^2}{4}}},$$

which depends only on $a - b$, and therefore, *the radius of curvature is the same at all tangent points of curves of the same system.*

At apses, $Y = 0$, and $s = 1$;

$$\therefore \frac{2ds}{s^3 dr} = -\frac{1}{X^2}.\frac{dY}{dr};$$

$$\therefore \frac{ds}{dr} = -\frac{1}{2X^2}.(a + b - 2r) = \frac{l}{2\left(k_{,} + k\dfrac{l^2}{4}\right)^2}, \text{ if } r = a,$$

$$\text{and } = -\frac{l}{2\left(k_{,} + k\dfrac{l^2}{4}\right)^2}, \text{ if } r = b.$$

Let R_a represent the radius of curvature, when $r = a$;

$$\therefore \frac{1}{R_a} = \frac{s}{r} + \frac{ds}{dr} = \frac{1}{a} + \frac{l}{2\left(k_{,} + k\dfrac{l^2}{4}\right)^2}$$

$$\frac{1}{R_b} = \frac{1}{b} - \frac{l}{2\left(k_{,} + k\dfrac{l^2}{4}\right)^2}. \qquad (6).$$

Hence also the following equations:

$$\frac{1}{R_a} + \frac{1}{R_b} = \frac{1}{a} + \frac{1}{b},$$

$$\frac{1}{R_{a_,}} + \frac{1}{R_{b_,}} = \frac{1}{a_,} + \frac{1}{b_,},$$

$$\frac{1}{R_{a_,}} + \frac{1}{R_b} = \frac{1}{a_,} + \frac{1}{b},$$

$$\frac{1}{R_a} + \frac{1}{R_{b_,}} = \frac{1}{a} + \frac{1}{b_,}.$$

Also, as $$\frac{1}{R_r} = \frac{s}{r} + \frac{ds}{dr},$$

and $$\frac{1}{R_{r_,}} = \frac{s}{r_,} + \frac{ds}{dr_,} = \frac{s}{r_,} - \frac{ds}{dr};$$

$$\therefore \frac{1}{R_r} + \frac{1}{R_{r_,}} = \frac{s}{r} + \frac{s}{r_,}.$$

If $f(r)$ be the radius of curvature of a curve at a tangent point; the radii of curvature when r becomes $r \mp h$, are

$$R = f(r \mp h) = f(r) \mp f'(r) . h,$$

and, the corresponding radii of curvature of another curve rolling with the former, are

$$R_, = \phi(r_, \pm h) = \phi(r_,) \pm \phi'(r_,) . h$$
$$= \phi(r_,) \mp \phi'(c - r) . h;$$

also, since the radii of curvature of the two curves at the tangent points are equal

$$f(r) = \phi(r_,);$$

and therefore $R - R_, = \mp \{f'(r) - \phi'(c - r)\} . h.$

Hence, if $R > R_,$ before the tangent points come into contact, $R < R_,$ afterwards, and consequently *the curves cross and change their rolling sides at the tangent points:* except $k_, = 0$, when there is a point of contrary flexure at the tangent points, which then also coincide at the mean distance.

If $r = \frac{a+b}{2}$, $\frac{ds}{dr} = 0$, or the curve makes a maximum or minimum angle with the radius vector at the mean distance; and the reciprocal of the radius of curvature

$$= \frac{k_{,}}{\frac{a+b}{2} \cdot \sqrt{k_{,}^2 + \left(\frac{a-b}{2}\right)^2}}.$$

The area of a wheel may be found: for the area of a lobe is the integral from $\phi = -\frac{\pi}{2}$ to $\phi = \frac{\pi}{2}$ of

$$r^2 d\theta = \left\{k_{,} + k \cdot \left(\frac{a-b}{2}\right)^2 \cdot \sin^2 \phi\right\} \cdot \left(\frac{a+b}{2} + \frac{a-b}{2} \cdot \sin \phi\right) \cdot d\phi,$$

$$\text{which } = \frac{a+b}{2} \cdot k_{,}\pi + k \cdot \frac{a+b}{16} \cdot (a-b)^2 \cdot \pi;$$

therefore, the area of a wheel

$$= \frac{a+b}{2} k_{,} n\pi + k \cdot \frac{a+b}{16} \cdot (a-b)^2 \cdot n\pi,$$

in which expression, if the value of $k_{,}$ be substituted from the equation

$$\frac{2k_{,}}{\sqrt{ab}} + k \cdot \frac{(a+b)^2}{2\sqrt{ab}} - k \cdot (a+b) = \frac{2}{n},$$

the area of a wheel

$$= \frac{\pi}{2} \cdot (a+b) \cdot \sqrt{ab} + \frac{kn\pi}{16} \cdot \{4 \cdot (a+b)^2 \cdot \sqrt{ab} - 2 \cdot (a+b)^3 + (a+b) \cdot (a-b)^2\},$$

$$= \frac{\pi}{2} \cdot (a+b) \cdot \sqrt{ab} + \frac{kn\pi}{16} \cdot (a+b) \cdot \{(a-b)^2 - 2 \cdot (a+b) \cdot (\sqrt{a} - \sqrt{b})^2\},$$

$$= \frac{\pi}{2} \cdot (a+b) \cdot \sqrt{ab} + \frac{kn\pi}{16} \cdot (a+b) \cdot (\sqrt{a} - \sqrt{b})^2 \cdot \{(\sqrt{a} + \sqrt{b})^2 - 2 \cdot (a+b)\},$$

$$= \frac{\pi}{2} \cdot (a+b) \cdot \sqrt{ab} - kn\pi \cdot (a+b) \cdot \left(\frac{\sqrt{a} - \sqrt{b}}{2}\right)^4.$$

Those systems of curves where $k = 0$, have no tangential points; for

$$\frac{d\theta}{dr} = \frac{k_{,}}{\sqrt{(a-r).(r-b)}},$$

and therefore cannot vanish.

If $k_{,} = 0$, there is always a tangential point in the middle of each half-lobe.

The former deserve a more particular consideration, as being in general more simple in form, and admitting of easy and elegant construction: if a_n, b_n be the major and minor apsidal distances of a wheel of n lobes, the equations of condition (2) are reduced to $a_n - b_n =$ constant $= l$,

$$\text{and } \frac{k_{,}}{\sqrt{a_n b_n}} = \frac{1}{n};$$

$$\text{and therefore, } a_n = \frac{l}{2} + \sqrt{n^2 k_{,}^2 + \frac{l^2}{4}},$$

$$\text{and } b_n = -\frac{l}{2} + \sqrt{n^2 k_{,}^2 + \frac{l^2}{4}},$$

and the equation to a curve of n lobes will then be

$$\theta = \frac{2}{n} \cdot \tan^{-1} \frac{\sqrt{n^2 k_{,}^2 + \frac{l^2}{4}} + \frac{l}{2}}{n k_{,}} \cdot \sqrt{\frac{r + \frac{l}{2} - \sqrt{n^2 k_{,}^2 + \frac{l^2}{4}}}{\frac{l}{2} - r + \sqrt{n^2 k_{,}^2 + \frac{l^2}{4}}}};$$

$$\text{or, } r = \frac{2 n^2 k_{,}^2}{2\sqrt{n^2 k_{,}^2 + \frac{l^2}{4}} + l . \cos\theta}.$$

Describe therefore a circle whose diameter is l, and draw (fig. 16) a tangent at any point A, in which take $AC = k_{,}$ and $AE = nk_{,}$ and draw EG through the centre: then the apsidal distances for a wheel of n lobes are EG and EF;

$$\text{for } EF = EO - FO = -\frac{l}{2} + \sqrt{n^2 k_{,}^2 + \frac{l^2}{4}} = b_n,$$

$$\text{and } EG = EO + OG = \frac{l}{2} + \sqrt{n^2 k_{,}^2 + \frac{l^2}{4}} = a_n.$$

Examples: if $k^2 = \frac{2}{9}$, and $n = 1, 3, 4$ &c., the figures (17), (18), (19), will roll together, or in pairs, and are also self-rolling curves.

The point of contrary flexure, when there is one, is always nearer to the centre than the mean distance: for if p be the perpendicular on the tangent from the centre,

$$\frac{rd\theta}{dr} = \frac{k_{,}}{\sqrt{(a-r).(r-b)}} = \frac{p}{\sqrt{r^2 - p^2}}, \text{ and if } dp = 0,$$

$$r = \frac{n^2 - 1}{n^2} \cdot \frac{2ab}{a+b}; \text{ also since } (a-b)^2 \text{ is positive,}$$

$$\frac{2ab}{a+b} < \cdot \frac{a+b}{2}$$

$$\text{and } \frac{n^2-1}{n^2} \text{ is an improper fraction;}$$

$$\therefore\ r < \frac{a+b}{2}.$$

Since $r - b$, which must be positive,

$$= \frac{b}{n^2.(a+b)} \cdot (n^2 l - 2a),$$

there is no point of contrary flexure, unless $n^2 l - 2a$ is positive.

The outline of the lobes may be traced without the use of logarithms by observing, since the equation in this case is

$$r = \frac{2ab}{a + b + (a-b).\cos n\theta},$$

that if $n\theta = \quad 0, \quad r = b,$

$$\frac{\pi}{12}, \quad r = \frac{4\sqrt{2}\,.\,ab}{a\,.\,(2\sqrt{2} + \sqrt{3} + 1) + b\,.\,(2\sqrt{2} - \sqrt{3} - 1)},$$

$$\frac{2\pi}{12}, \quad r = \frac{4ab}{2\,.\,(a + b) + (a - b)\,.\,\sqrt{3}},$$

$$\frac{3\pi}{12}, \quad r = \frac{2\sqrt{2}\,.\,ab}{a\,.\,(1 + \sqrt{2}) + b\,.\,(\sqrt{2} - 1)},$$

$$\frac{4\pi}{12}, \quad r = \frac{4ab}{3a + b},$$

$$\frac{5\pi}{12}, \quad r = \frac{4\sqrt{2}\,.\,ab}{a\,.\,(2\sqrt{2} + \sqrt{3} - 1) + b\,.\,(2\sqrt{2} + 1 - \sqrt{3})},$$

$$\frac{6\pi}{12}, \quad r = \frac{2ab}{a + b},$$

$$\frac{7\pi}{12}, \quad r = \frac{4\sqrt{2}\,.\,ab}{a\,.\,(2\sqrt{2} + 1 - \sqrt{3}) + b\,.\,(2\sqrt{2} + \sqrt{3} - 1)},$$

$$\frac{8\pi}{12}, \quad r = \frac{4ab}{a + 3b},$$

$$\frac{9\pi}{12}, \quad r = \frac{2\sqrt{2}\,.\,ab}{a\,.\,(\sqrt{2} - 1) + b\,.\,(\sqrt{2} + 1)},$$

$$\frac{10\pi}{12}, \quad r = \frac{4ab}{2\,.\,(a + b) - (a - b)\,.\,\sqrt{3}},$$

$$\frac{11\pi}{12}, \quad r = \frac{4\sqrt{2}\,.\,ab}{a\,.\,(2\sqrt{2} - \sqrt{3} - 1) + b\,.\,(2\sqrt{2} + \sqrt{3} + 1)},$$

$$\pi, \quad r = a.$$

Hence the following rule: Describe the circle whose radius is the minor distance, and divide it into n equal parts, each of which will form the base of a lobe; divide half the base into twelve equal parts, and draw straight lines from the centre, through the points of division, respectively equal to the above values: and the curve drawn through their extremities will be the outline of half a lobe (fig. 20).

The distances may also be found practically, by describing an ellipse whose axis major is $a_n + b_n$, and $a_n - b_n$ the distance between its foci; then if straight lines be drawn from one of the foci to the ellipse making equal angles with each other, and the base of the lobe be divided into as many equal parts as there are equal angles round the focus: the distances from the centre to the several points of the lobe are easily shewn to be equal to the elliptic distances; and may therefore be set off from them.

The form of a *rack*, or curve of an infinite number of lobes to move with the curves derived from the equation

$$\theta = \left\{\frac{2k_{,}}{\sqrt{ab}} + \frac{k.(a+b)^2}{2\sqrt{ab}}\right\}.\tan^{-1}\sqrt{\frac{a}{b}}.\sqrt{\frac{r-b}{a-r}}$$

$$- k.\sqrt{(a-r).(r-b)} - k.(a+b).\tan^{-1}\sqrt{\frac{r-b}{a-r}},$$

may be found by making n infinite and $a - b = l$, where a and b are also infinite; and this form is that to which the lobes gradually approach as n increases: if x and y be rectangular co-ordinates of the rack, x being measured along its base from one of the apses, and y be perpendicular to the base, $x = b\theta$ and $y = r - b$;

$$\therefore\ x = \left\{2k_{,}\sqrt{\frac{b}{a}} + \frac{k}{2}\sqrt{\frac{b}{a}}.(a+b)^2\right\}.\tan^{-1}\sqrt{\frac{a}{b}}.\sqrt{\frac{y}{l-y}}$$

$$- bk\sqrt{ly - y^2} - k.(ab + b^2).\tan^{-1}\sqrt{\frac{y}{l-y}}.$$

By Maclaurin's Theorem, the expansion of $\tan^{-1}\sqrt{\frac{a}{b}}.\sqrt{\frac{y}{l-y}}$, or of $\tan^{-1}\left(1+\frac{l}{b}\right)^{\frac{1}{2}}.\sqrt{\frac{y}{l-y}}$ as far as the square of $\frac{l}{b}$ is

$$\tan^{-1}\sqrt{\frac{y}{l-y}} + \frac{1}{2b}\sqrt{ly-y^2} - \frac{l+2y}{8b^2}.\sqrt{ly-y^2},$$

and $\frac{(a+b)^2}{2}.\sqrt{\frac{b}{a}} = \left(2b^2 + 2bl + \frac{l^2}{2}\right).\left(1 - \frac{l}{2b} + \frac{3l^2}{8b^2}\right)$

$= 2b^2 + bl + \frac{l^2}{4}$, omitting the negative powers of b, as b is infinite;

$$\therefore \frac{(a+b)^2}{2} \cdot \sqrt{\frac{b}{a}} \cdot \tan^{-1} \sqrt{\frac{b}{a}} \cdot \sqrt{\frac{y}{l-y}}$$

$$= \left(2b^2 + bl + \frac{l^2}{4}\right) \cdot \tan^{-1} \sqrt{\frac{y}{l-y}} + \left(b + \frac{l}{2}\right) \cdot \sqrt{ly - y^2};$$

$$\text{also } b \cdot \sqrt{ly - y^2} + b \cdot (a+b) \cdot \tan^{-1} \sqrt{\frac{y}{l-y}}$$

$$= b\sqrt{ly - y^2} + (2b^2 + bl) \cdot \tan^{-1} \sqrt{\frac{y}{l-y}},$$

which quantities being substituted in the above equation, we have the equation to the rack

$$x = \left(2k_, + \frac{kl^2}{4}\right) \cdot \tan^{-1} \sqrt{\frac{y}{l-y}} + k \cdot \frac{l-2y}{4} \cdot \sqrt{ly - y^2};$$

from which it appears that each lobe of the rack is composed of four similar and equal parts.

This equation may also be found from the differential equation

$$\frac{dx}{dy} = \frac{k_, + k \cdot \left(y - \frac{l}{2}\right)^2}{\sqrt{ly - y^2}},$$

which is immediately deducible from

$$\frac{rd\theta}{dr} = \frac{k_, + k \cdot \left(r - \frac{a+b}{2}\right)^2}{\sqrt{(a-r) \cdot (r-b)}}.$$

If $k = 0$, $y = l \cdot \sin^2 \frac{x}{2k}$ (fig. 21), which is a rack that will roll with figures (17), (18), (19).

HAMNETT HOLDITCH.

NOTE ON FRICTION WHEELS.

It was observed, that a rolling *continuous* curve cannot drive another after the driving point has reached its maximum distance: if, however, the curves are discontinuous, and a new driving point shall come into action at the moment the former driving point shall have reached its maximum distance, a continued revolving motion without friction, may, under certain circumstances be produced; and this will be the case if two wheels be formed of semilobes of the *same system*, if clogging of the wheels can be avoided; for (fig. 23) when the driving point of *A* has arrived at *G*, a new driving point will come into action at *B*.

The variation of the angular velocity of the wheel driven, supposing that of the driving wheel to be uniform; the oblique mechanical action of the driving wheel near the apses, which at the apses is towards the centre, and the shocks produced at the change of the driving points, which would however be received at the flat surfaces, would unfit such wheels for the purpose of moving weights; it may still be a question, whether they might not be successfully employed for purposes of motion.

When the new driving point comes into action, it is necessary that the point *F* should clear itself of the point *G*. The relative motions of the wheels will be the same if the wheel *B* be supposed at rest, and the other to move round it; and therefore the point *F* must describe a curve without the wheel *B*, or the radius of curvature of the curve described by the point *F*, immediately after the change of the driving points, must be less than the radius of curvature at *G*, supposing the curvature at *G* to be convex towards the centre of *B*; in which case, the wheels will not clog at *G*, when the driving point is changed.

Let *R* be the radius of curvature described by *F*; if the wheel *A* be supposed to have moved a little, the motion of *F* will be perpendicular to *FC*; and *GH*, *FH* being consecutive normals, *FH* will be the radius of curvature of the curve described by *F*, *CD* the radius of curvature of the wheel *B* at the major apse, and *CE* that of *A* at the minor apse.

Let the small angle $BDC = \theta$; $\therefore\ \angle\ CEA = \theta \cdot \frac{CD}{CE}$

$$\angle\ CFA = \angle\ CEA \cdot \frac{CE}{CF} = \theta \cdot \frac{CD}{CF};$$

$$\therefore\ \angle\ FCE = \theta \cdot \left(\frac{CD}{CE} - \frac{CD}{CF}\right),$$

$$\text{and } HD = HC \cdot \frac{\sin C}{\sin D} = HC \cdot \left(\frac{CD}{CE} - \frac{CD}{CF}\right);$$

or, if a and b be the major and minor distances of the wheel B, and R_a, R_b represent the radii of curvature at B and G, and similar quantities with dashes those of the other wheel, and $a - b = l$; then

$$l - R - R_a = (l - R) \cdot R_a \cdot \left(\frac{1}{R_{b_i}} - \frac{1}{l}\right);$$

$$\text{and therefore, } R = \frac{l^2 \cdot (R_{b_i} - R_a)}{l \cdot (R_{b_i} - R_a) + R_a \cdot R_{b_i}}.$$

To prevent clogging at G, therefore $R < -R_b$,

$$\text{or } -\frac{1}{R_b} < \frac{1}{R};$$

$$\therefore \text{ by equation (6) } \frac{l}{2\left(k_i^2 + k\frac{l^2}{4}\right)^2} - \frac{1}{b} < \frac{1}{l} + \frac{R_a R_{b_i}}{l^2 (R_{b_i} - R_a)}$$

$$< \frac{1}{l} + \frac{1}{l^2\left(\frac{1}{R_a} - \frac{1}{R_{b_i}}\right)}.$$

$$\text{Now } \frac{1}{R_a} = \frac{1}{a} + \frac{l}{2\left(k_i + k\frac{l^2}{4}\right)^2},$$

$$\text{and } \frac{1}{R_{b_i}} = \frac{l}{2\left(k_i + k\frac{l^2}{4}\right)^2} - \frac{1}{b_i},$$

the curves being considered convex to the centres at the minor apses;

$$\therefore \frac{1}{R_a} - \frac{1}{R_{b_{,}}} = \frac{1}{a} + \frac{1}{b_{,}}, \text{ and}$$

$$\therefore \frac{l}{2\left(k_{,}^2 + k\frac{l^2}{4}\right)^2} - \frac{1}{b} < \frac{1}{l} + \frac{1}{l^2 \cdot \left(\frac{1}{a} + \frac{1}{b_{,}}\right)}.$$

It may be shewn in the same way, in order that the wheels may not clog at the point *B* *before* the driving point at *A* comes into action, that

$$\frac{l}{2\left(k_{,}^2 + k\frac{l^2}{4}\right)^2} - \frac{1}{b_{,}} < \frac{1}{l} + \frac{1}{l^2 \cdot \left(\frac{1}{a_{,}} + \frac{1}{b}\right)},$$

and as a, b, $a_{,}$, $b_{,}$ must be positive quantities, both these conditions will be fulfilled, if

$$\frac{l}{2\left(k_{,} + k\frac{l^2}{4}\right)^2} = \frac{1}{l};$$

$$\text{or, } k_{,} + k \cdot \frac{l^2}{4} = \frac{l}{\sqrt{2}}, \qquad (7),$$

which may be called the *clearing equation;* if the value of $k_{,}$ from this be substituted in the equation

$$\frac{2k_{,}}{\sqrt{ab}} + \frac{k \cdot (a+b)^2}{2\sqrt{ab}} - k \cdot (a+b) = \frac{2}{n},$$

we have finally

$$\frac{l\sqrt{2}}{\sqrt{ab}} - k \cdot (\sqrt{a} - \sqrt{b})^2 = \frac{2}{n}, \qquad (8),$$

for determining the radii of a friction wheel of $2n$ teeth; and by giving different values to n, sets of friction wheels will be found which will not clog theoretically just *before* or *after* the change of the driving teeth: and such wheels will not clog at other points, unless the depth of the teeth be very great in proportion to the radii of the wheels, or the curves used for the construction of the teeth be of complicated forms.

An example is given in figure (24), where $k = 0$, and the clearing equation (7) becomes $2k_{,}^{2} = l^2$, and equation (8) for determining the radii is therefore

$$\sqrt{2ab} = nl, \quad \text{and } a = \frac{l}{2}.(1 + \sqrt{2n^2+1}),$$

$$b = \frac{l}{2}.(-1 + \sqrt{2n^2+1}).$$

Hence, for a wheel of eight teeth, which is derived from a curve of four lobes,

$$n = 4, \quad \text{and } \left.\begin{array}{l} a = 3{\cdot}37 \\ b = 2{\cdot}37 \end{array}\right\}, \text{ if } l = 1.$$

$$\text{If } n = 6, \quad \left.\begin{array}{l} a_{,} = 4{\cdot}77 \\ b_{,} = 3{\cdot}77 \end{array}\right\},$$

for a wheel of twelve teeth to turn with the former, and the teeth (or half-lobes) may be described from rules before given.

The flat sides of the teeth must be a little hollowed out to allow of the free motion of the points, but these have no connection with the rolling sides.

IV. *Note on the Motion of Waves in Canals.* *By* G. Green, Esq. B.A. *of Caius College.*

[Read *February* 18, 1839.]

In a former communication I have endeavoured to apply the ordinary Theory of Fluid Motion to determine the law of the propagation of waves in a rectangular canal, supposing ζ the depression of the actual surface of the fluid below that of equilibrium very small compared with its depth; the depth γ as well as the breadth β of the canal being small compared with the length of a wave. For greater generality, β and γ are supposed to vary very slowly as the horizontal co-ordinate x increases, compared with the rate of the variation of ζ, due to the same cause. These suppositions are not always satisfied in the propagation of the tidal wave, but in many other cases of propagation of what Mr Russel denominates the "Great Primary Wave," they are so, and his results will be found to agree very closely with our theoretical deductions. In fact, in my paper on the Motion of Waves, it has been shown that the height of a wave varies as

$$\beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}.$$

With regard to the effect of the breadth β, this is expressly stated by Mr Russel (Vide Seventh Report of the British Association, p. 425), and the results given in the tables, p. 494, of the same work, seem to agree with our formula as well as could be expected, considering the object of the experiments there detailed.

In order to examine more particularly the way in which the Primary Wave is propagated, let us resume the formulæ,

$$\phi = \beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}} F\left(t - \int \frac{dx}{\sqrt{g\gamma}}\right),$$

$$\zeta = \frac{d\phi}{g dt} = \frac{\beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}}{g} F'\left(t - \int \frac{\delta x}{\sqrt{g\gamma}}\right);$$

where we have neglected the function f, which relates to the wave propagated in the direction of x negative.

Suppose, for greater simplicity, that β and γ are constant, the origin of x being taken at the point where the wave commences when $t = 0$. Then we may, without altering in the slightest degree the nature of our formulæ, take the values,

$$(1) \qquad \phi = F(x - t\sqrt{g\gamma}),$$

$$\zeta = \frac{d\phi}{g dt} = -\sqrt{\frac{\gamma}{g}} . F'(x - t\sqrt{g\gamma}).$$

But for all small oscillations of a fluid, if (a, b, c) are the co-ordinates of any particle P in its primitive state, that of equilibrium suppose; (x, y, z) the co-ordinates of P at the end of the time t, and $\Phi = \int \phi dt$ when (x, y, z) are changed into (a, b, c), we have (Vide *Mécanique Analytique*, Tome II. p. 313.)

$$(2) \qquad x = a + \frac{d\Phi}{da}, \qquad y = b + \frac{d\Phi}{db}, \qquad z = c + \frac{d\Phi}{dc}.$$

Applying these general expressions to the formulæ (1) we get

$$\Phi = -\frac{1}{\sqrt{g\gamma}} {}'F(a - t\sqrt{g\gamma}), \quad \text{and } x = a - \frac{1}{\sqrt{g\gamma}} F(a - t\sqrt{g\gamma}).$$

Neglecting (disturbance)², we have

$$\zeta = -\sqrt{\frac{\gamma}{g}} F'(a - t\sqrt{g\gamma}),$$

and consequently,

$$\int_0^a \zeta(a - t\sqrt{g\gamma}) \, . \, da = -\sqrt{\frac{\gamma}{g}} \, F(a - t\sqrt{g\gamma}),$$

supposing for greater simplicity that the origin of the integral is at $a = 0$.

Hence the value of x becomes

$$x = a + \frac{1}{\gamma} \int_0^a da \zeta(a - t\sqrt{g\gamma}).$$

Suppose α = length of the wave when $t = 0$; then $\zeta(a) = 0$, except when a is between the limits 0 and α. If therefore we consider a point P before the wave has reached it,

$$\int_0^a da \zeta(a - t\sqrt{g\gamma}) = \int_0^a da \zeta(a) = V;$$

the whole volume of the fluid which would be required to fill the hollow caused by the depression ζ below the surface of equilibrium when $t = 0$. Hence we get

$$x' = a + \frac{V}{\gamma};$$

x' being the horizontal co-ordinate of P, before the wave reaches P.

Also, let x'' be the value of this co-ordinate after the wave has passed completely over P, then

$$\int_0^a da \zeta(a - t\sqrt{g\gamma}) = 0, \quad \text{and } x'' = a.$$

If ζ were wholly negative, or the wave were elevated above the surface of equilibrium, we should only have to write $-V$ for V, and thus

$$x' = a - \frac{V}{\gamma}, \quad \text{and } x'' = a.$$

We see therefore, in this case, that the particles of the fluid by the transit of the wave are transferred forwards in the direction of the wave's motion, and permanently deposited at rest in a new place at some distance from their original position, and that the extent of the transference is sensibly equal throughout the whole depth. These waves are called by Mr Russel, positive ones, and this result agrees with his experiments, Vide p. 423. If however ζ were positive, or the wave wholly depressed, it follows from our formula, that the transit of the fluid particles would be in the opposite direction. The experimental investigation of those waves, called by Mr Russel, negative ones, has not yet been completed, p. 445, and the last result cannot therefore be compared with experiment.

The value $\frac{V}{\gamma}$ which we have obtained analytically for the extent over which the fluid particles are transferred, suggests a simple physical reason for the fact. For previous to the transit of a positive wave over any particle P, a volume of fluid behind P, and equal to V, is elevated above the surface of equilibrium. During the transit, this descends within the surface of equilibrium, and must therefore force the fluid about P forward through the space

$$\left(\frac{V}{\gamma}\right);$$

admitting as an experimental fact, that after the transit of the wave the fluid particles always remain absolutely at rest.

Mr Russel, p. 425, is inclined to infer from his experiments, that the velocity of the Great Primary Wave is that due to gravity acting through a height equal to the depth of the centre of gravity of the transverse section of the channel below the surface of the fluid. When this section is a triangle of which one side is vertical, as in Channel (H), p. 443, the ordinary Theory of Fluid Motion may be applied with extreme facility. For if we take the lowest edge of the horizontal channel as the axis of x, and the axis of z vertical and directed upwards, the general equations for small oscillations in this case become

$$(A) \qquad 0 = gz + \frac{p}{\rho} + \frac{d\phi}{dt},$$

$$(B) \qquad 0 = \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2},$$

we have, also, the conditions

$$(a) \qquad v = \frac{d\phi}{dy} = 0 \text{ (when } y = 0\text{)},$$

$$(b) \qquad \frac{w}{v} = \frac{\frac{d\phi}{dz}}{\frac{d\phi}{dy}} = \frac{z}{y} \text{ (when } \frac{z}{y} = \cot \alpha\text{)},$$

α being the angle which the inclined side of the channel makes with the vertical.

The first of these conditions is due to the vertical side, and the second to the inclined one, since at these extreme limits the fluid particles must move along the sides.

Now from what has been shown in our memoir, it is clear that we may satisfy the equation (B) and the two conditions just given, by

$$(c) \qquad \phi = \phi_0 + \phi_{,}(y^2 + z^2),$$

ϕ and $\phi_{,}$ being two such functions of x and t only that

$$(C) \qquad 0 = \frac{d^2\phi_0}{dx^2} + 4\phi_{,}.$$

It now only remains to satisfy the condition due to the upper surface. Let therefore

$$0 = z - \zeta_{x \cdot t}$$

be the equation of this surface. Then the formula (A) of our paper before cited gives

$$0 = \frac{d\phi}{dz} - \frac{d\zeta}{dt} - \frac{d\zeta}{dx}\frac{d\phi}{dx} \text{ (when } z = c + \zeta\text{)}$$

or neglecting (disturbance)²

$$0 = \frac{d\phi}{dz} - \frac{d\zeta}{dt} \text{ (when } z = c\text{)};$$

c being the vertical depth of the fluid in equilibrium.

Also at the upper surface $p = 0$, therefore by continuing to neglect (disturbance)² (A) gives

$$0 = g\zeta + \frac{d\phi}{dt} \text{ (when } z = c\text{)}.$$

Hence, by eliminating ζ, we get

$$0 = g\frac{d\phi}{dz} + \frac{d^2\phi}{dt^2} \text{ (when } z = c\text{)},$$

which by (c) becomes, when we neglect terms of the order y^2 and z^2 compared with those retained,

$$0 = 2gc\phi_{,} + \frac{d^2\phi_0}{dt^2}.$$

Or eliminating $\phi_{,}$ by means of (C),

$$0 = \frac{d^2\phi_0}{dt^2} - \frac{gc}{2}\cdot\frac{d^2\phi_0}{dx^2}.$$

The particular integral of which belonging to the wave that proceeds in the direction of x positive is

$$\phi_0 = f\left(x - t\sqrt{\frac{gc}{2}}\right),$$

and hence the velocity of propagation of the wave is

$$(D) \qquad v' = \sqrt{\frac{gc}{2}}.$$

Mr Russel gives $\sqrt{\frac{2gc}{3}}$ as the velocity, but at the same time remarks, that in consequence of the attraction of the sides of the canal fixing a portion of the fluid in its lower angle, we ought in employing any formula to calculate for an *effective* depth in place of the real one, p. 442. Instead of adopting this method, let us compare the formula (D) given by the common Theory of Fluid Motion, with Mr Russel's experiments. And as in our theory we have considered those waves only in which the elevation above the surface of equilibrium is very small compared with the depth c, it will be necessary to select those waves in which this condition is nearly satisfied. I have therefore taken from the Table, p. 443, all the waves in which

$$\zeta < \frac{c}{20},$$

and have supposed $g = 32\frac{1}{6}$ feet: the results are given below.

Observation.	Value of c.	Observed Vel. viz. feet per second.	Velocity by formula (D).
lviii	4, in.	2,19	2,313
lxii	5,11	2,58	2,617
lxvi	6,04	2,85	2,845
lxvii	6,05	2,88	2,847
lxxv	7,04	3,03	3,072
lxxii	7,04	3,05	3,072
lxxiii	7,04	3,04	3,072
lxxi	7,04	3,02	3,072
lxxiii	7,04	3,02	3,072

A more perfect agreement with theory than this could scarcely be expected. Had the formula $\sqrt{\frac{2gc}{3}} = v$ been used, the errors would have been much greater.

The theory of the motion of waves in a deep sea, taking the most simple case, in which the oscillations follow the law of the cycloidal pendulum, and considering the depth as infinite, is extremely easy, and may be thus exhibited.

Take the plane (xz) perpendicular to the ridge of one of the waves supposed to extend indefinitely in the direction of the axis y, and let the velocities of the fluid particles be independent of the co-ordinate y. Then if we conceive the axis z to be directed vertically downwards, and the plane (xy) to coincide with the surface of the sea in equilibrium, we have generally,

$$gz - \frac{p}{\rho} = \frac{d\phi}{dt},$$

$$0 = \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dz^2}.$$

The condition due to the upper surface, found as before, is

$$0 = g\frac{d\phi}{dz} - \frac{d^2\phi}{dt^2}.$$

From what precedes, it will be clear that we have now only to satisfy the second of the general equations in conjunction with the condition just given. This may be effected most conveniently by taking

$$\phi = H\epsilon^{-\frac{2\pi}{\lambda}z} \sin\frac{2\pi}{\lambda}(v't - x),$$

by which the general equation is immediately satisfied, and the condition due to the surface gives

$$g = \frac{2\pi}{\lambda}v'^2, \quad \text{or } v' = \sqrt{\frac{g\lambda}{2\pi}},$$

where λ is evidently the length of a wave. Hence, the velocity of these waves vary as $\sqrt{\lambda}$, agreeably to what Newton asserts. But the velocity assigned by the correct theory exceeds Newton's value in the ratio $\sqrt{\pi}$ to $\sqrt{2}$, or of 5 to 4 nearly.

What immediately precedes is not given as new, but merely on account of the extreme simplicity of the analysis employed. We shall, moreover, be able thence to deduce a singular consequence which has not before been noticed, that I am aware of.

Let (a, b, c) be the co-ordinates of any particle P of the fluid when in equilibrium. Then, since

$$\phi = H\epsilon^{-\frac{2\pi}{\lambda}z} \sin \frac{2\pi}{\lambda}(v't - x); \qquad \therefore\ \Phi = \frac{-H\lambda}{2\pi v'}\epsilon^{-\frac{2\pi c}{\lambda}} \cos \frac{2\pi}{\lambda}(v't - a),$$

and the general formulæ (2) give

$$x = a + \frac{d\Phi}{da} = a - \frac{H}{v'}\epsilon^{-\frac{2\pi c}{\lambda}} \sin \frac{2\pi}{\lambda}(v't - a),$$

$$z = c + \frac{d\Phi}{dc} = c + \frac{H}{v'}\epsilon^{-\frac{2\pi c}{\lambda}} \cos \frac{2\pi}{\lambda}(v't - a).$$

Hence,

$$(x - a)^2 + (y - b)^2 = \left(\frac{H}{v'}\epsilon^{-\frac{2\pi c}{\lambda}}\right)^2,$$

and therefore any particle P revolves continually in a circular orbit, of which the radius is

$$\frac{H}{v'}\epsilon^{-\frac{2\pi}{\lambda}c},$$

round the point which it would occupy in a state of equilibrium. The radius of this circle, and consequently the agitation of the fluid particles, decreases very rapidly as the depth c increases, and much more rapidly for short than long waves, agreeably to observation.

Moreover, the direction of the rotation is such, that in the upper part of the circle the point P moves in the direction of the motion of the wave. Hence, as in the propagation of the Great Primary Wave, the actual motion of the fluid particles is direct where the surface of the fluid rises above that of equilibrium, and retrograde in the contrary case.

V. *On the Nature of the Molecular Forces which regulate the Constitution of the Luminiferous Ether.* *By* S. Earnshaw, M.A. *of St. John's College, Cambridge.*

[Read *March* 18, 1839.]

There are already before the world by various authors several Memoirs, which, collaterally or incidentally, embrace the subject of the present communication. There is observable in them, however, much disagreement of results, which seems chiefly to arise from the extreme length and complexity of the investigations by which those results are obtained; to avoid which, as much as possible, their authors are compelled to adopt means of simplification, which we cannot always be certain *à priori* are sufficiently approximative. In the following pages the subject will be found to be treated in a manner perfectly new and direct, and, it is hoped also, satisfactory, inasmuch as the analytical operations employed are brief and simple, involving no principles of a difficult or doubtful character.

The authors to which I have just alluded have generally adopted, as a most extensive means of simplification, symmetrical arrangements of the particles of the ethereal medium. This may be necessary and even allowable in some cases: but as it has never been shewn that such arrangements actually *do exist* in Nature, nor even that they *can exist* in Nature, I have been careful to confine myself to the investigation of properties which are independent of arrangement, or rather, which do not involve the hypothesis of a peculiar arrangement.

I may also remark that the investigations which follow are in other respects of a very general character. For, in this attempt to discover the laws of molecular action of the ether, amongst the experimental properties, assumed as the basis of analytical investigation, are, I believe, none which are *peculiar* to the luminiferous ether. I think it probable, that most terrestrial bodies possess in a greater or less degree of perfection the properties here assumed: and consequently, the title of this paper might have been made more comprehensive. It might, perhaps, not improperly be, "*An Investigation of the Nature of the Molecular Forces, which regulate the Internal Constitution of Bodies.*" This might, however, be disputed, and therefore in the investigations I have referred only to the luminiferous ether. Nevertheless, that the reader may more easily judge what degree of claim the following pages have to that general character which is here ascribed to them, I shall, in as few words as possible, introduce a statement of the experimental assumptions, and the results respectively derived from them.

I. It is assumed that the ether consists of detached particles; each of which is in a position of equilibrium, and when slightly disturbed is capable of *vibrating in any direction.* (Many solid as well as aerial bodies transmit sound, which is generally supposed to imply the existence of the same properties in them as are here assumed to be true of the ether.)

The most curious and perhaps least expected result of this assumption is, *that the molecular forces which regulate the vibrations of the ether do not vary according to Newton's law of universal gravitation:* and it is not a little remarkable, that a force, whether attractive or repulsive, varying according to this law, is the only one which *cannot possibly actuate* the particles of a *vibrating* medium.

II. It is next assumed, that the motion of a vibrating particle is more affected by the influence of the particles which are near to it than of those which are more remote. (This is certainly true of many other substances besides the ether.) The result which is sought to be derived from this assumption is, *that the molecular forces which regulate the vibra-*

tion of the particles are REPULSIVE, *and vary according to an inverse power of the distance greater than* 2.

III. It is lastly assumed, that the ether exists (or at least is capable of existing) as one mass held together by the *attraction* of its elementary molecules. This assumption is necessary, in order that the dispersion of the medium which would naturally result from the *repulsive* forces which regulate the vibration of its particles, may be thereby prevented.

The result which is derived from this necessary assumption is, *that each particle exerts* (in addition to the repulsive force before mentioned) *an attractive force, which varies according to Newton's Law of universal gravitation.*

By reversing the problem, I have been able to shew, that though Newton's law is the only one which cannot enable the particles to *vibrate*, yet it is the only law of force which can enable them to constitute and maintain themselves a *permanent* medium, without endangering, or in any way affecting their *vibrating* or luminiferous property.

I have on these grounds not hesitated to express my opinion, *that the particles of the luminiferous ether are each endued with two forces of distinct characters and uses; one attractive, to preserve themselves a permanent medium, varying inversely as the square of the distance; and the other repulsive, to which is due their luminiferous property, varying in a higher inverse ratio of the distance than the square.*

A SYSTEM OF DETACHED PARTICLES.

1. *If* V *denote the sum of the quotients formed by dividing each attracting body by its distance from the attracted body; then* $V = C$ *is the equation of a surface at any point of which if the attracted body be placed, it will begin to move in the direction of a normal.*

For, let f, g, h be the co-ordinates of the attracted body; F, G, H the attractions of the whole system upon it parallel to the co-ordinate axes, then

$$F = d_f V, G = d_g V, H = d_h V.$$

But the equation of the tangent plane at that point of the proposed surface where the attracted body is placed, satisfies the differential equation,

$$0 = d_f V . df + d_g V . dg + d_h V . dh;$$

$$\therefore\ 0 = F . df + G . dg + H . dh.$$

This equation shews that the resolved part of the attractive force is zero, in the direction of the tangent plane; and therefore the whole attraction is in the direction of the normal.

2. For the sake of brevity, I shall denominate the surface $V = C$, the *parametric* surface passing through the point f, g, h.

Different points in space may have corresponding different parametric surfaces; any one may be found by assigning the proper value to C. Their equations differ only in the value of the constant C, which, for this reason, I shall call the parameter.

If any parametric surface pass through an attracting particle, its parameter will be infinite, because at that point V is infinite; in which case the proposition will fail. The proposition is true of repulsive forces, or if some of the particles exert repulsive and some attractive forces; but when the forces are all attractive, V can neither be evanescent nor negative: since, however, it is infinite when the attracted particle touches any one of the attracting particles, and is not infinite in other positions, there must be some intermediate positions which make V a minimum, and there may be positions in which V is a maximum.

3. *The parametric surfaces which pass through points indefinitely near to a point of neutral attraction, are in general similar concentric*

hyperboloids of one and two sheets, the common centre of which is the point of neutral attraction. Certain points, however, have the asymptotic surface for their characteristic surface.

Let fgh be the co-ordinates of K, the point of neutral attraction, and $f + x$, $g + y$, $h + z$, the co-ordinates of P, a point very near to K. Let the value of V at K be C', and at P, C. Then the equations to the respective surfaces are

$$C' = V, \text{ and } C = V';$$

where V' is the same function of $f + x$, $g + y$, $h + z$ that V is of f, g, h.

$$\therefore \; C = V + d_f V . x + d_g V . y + d_h V . z + d_f^2 V . \frac{x^2}{2} + d_g^2 V . \frac{y^2}{2}$$

$$+ d_h^2 V . \frac{z^2}{2} + d_f d_g V . xy + d_f d_h V . xz + d_g d_h V . yz + \&c.$$

But because K is a point of neutral attraction, $d_f V = 0$, $d_g V = 0$, $d_h V = 0$, and

$$\therefore \; 2(C - C') = d_f^2 V . x^2 + d_g^2 V . y^2 + d_h^2 V . z^2 + 2 d_f d_g U . xy + \&c.$$

This, neglecting terms above the second order, being the general equation of surfaces of the second order which have a centre, by transposing the co-ordinate axes so as to coincide with the principal axes of the surface, the terms containing xy, yz, xz will disappear, leaving only

$$2(C - C') = d_f^2 V . x^2 + d_g^2 V . y^2 + d_h^2 V . z^2,$$

which for indefinitely small values of x, y, z may be regarded as the equation of the parametric surface. It must be remembered that the coefficients of x^2, y^2, z^2 are subject to the condition,

$$0 = d_f^2 V + d_g^2 V + d_h^2 V;$$

and because at least one of these coefficients will be negative, and one positive, the equation is that of an hyperboloid.

4. That parametric surface which contains a point of neutral attraction will be a cone, which is asymptotic to all the hyperbolic parametric surfaces belonging to the other points.

For, the parameter of the surface passing through K, the point of neutral attraction is C', and therefore the equation of it is

$$0 = d_f^2 V . x^2 + d_g^2 V . y^2 + d_h^2 V . z^2,$$

which is the asymptote of the surfaces included in the equation,

$$2(C - C') = d_f^2 V . x^2 + d_g^2 V . y^2 + d_h^2 V . z^2.$$

6. If the position of equilibrium be such, that only one of the quantities $d_f^2 V$, $d_g^2 V$, $d_h^2 V$ is negative, as for instance, $d_f^2 V$; then the axis of the asymptotic cone will coincide with the axis of x; and all points within this cone will have hyperboloids of one sheet for their parametric surfaces, and their parameters will be less than C'. The points without this cone will have hyperboloids of two sheets for their parametric surfaces, and their parameters will be greater than C'.

If the position of equilibrium be such, that two of the quantities $d_f^2 V$, $d_g^2 V$, $d_h^2 V$ are negative, as for instance, $d_g^2 V$ and $d_h^2 V$, the axis of the asymptotic cone and the parametric surfaces will be as in the last case; but the parameters of points within the cone will be greater than C', and of points without it, less than C'.

7. If the molecular forces are all repulsive, then the sign of V will be changed: but the parametric surfaces will be hyperboloids, as before.

8. If the position of equilibrium be such, that $d_f^2 V = 0$, $d_g^2 V = 0$, and $d_h^2 V = 0$, then $d_f V$, $d_g V$, $d_h V$, *i.e.* the attractions F, G, H, being also evanescent, the particle is unattracted in every direction, at least for small displacements from the position of equilibrium. An example of this is afforded in the case of a particle placed within a spherical or ellipsoidal surface, composed of attracting or repelling par-

ticles. If the position of equilibrium be such, that one of them, as $d_f^2 V$, is evanescent, then, F being evanescent also, for small displacements parallel to the axis of x, the particle will be unattracted. An example of this is afforded in the case of a particle placed within a hollow elliptic or circular cylinder of indefinite length. The displacement of particles placed in such positions as those here considered would not bring into action any forces of restoration; on which account the particles would not vibrate. It is evident, therefore, that the phenomena of light and sound are not due to the motions of particles placed in such positions: and as the purpose of this paper is to examine the constitution of media supposed to be capable of transmitting light, a phenomenon due to *vibration*, we shall, in what follows, always suppose that none of the quantities $d_f^2 V$, $d_g^2 V$, $d_h^2 V$ are evanescent: under which supposition also, they cannot be equal, since their sum $= 0$.

9. Since the force which urges a displaced particle acts in the direction of a normal to the parametric surface in which the particle is at any moment situated, there are in general only three directions in which a particle can be displaced, so that the force called into play may act in the direction of the displacement. These directions coincide with the principal axes of the parametric hyperboloids. The exceptions to this are, when the constitution of the system is such, that two of the three quantities, $d_f^2 V$, $d_g^2 V$, $d_h^2 V$, are equal; in which case the asymptotic cone has a circular base, and the exterior parametric hyperboloid becomes the hyperboloid of revolution of one sheet: and since the normals to this surface, corresponding to points in that principal section which is perpendicular to the axis of revolution, all pass through the centre, the force of restitution will always act in the line of displacement, when the particle is disturbed in any direction in this plane. This is the only exception.

10. It is very important to remark, that since the parametric surfaces cannot be spherical in any case, the constitution of a medium, composed of detached attractive particles, can never be such that the force of restitution called into play by a disturbance in *any* direction

shall act in the line of displacement. Hence those media which are distinguished as *uncrystallized*, cannot consist of detached particles which either attract or repel each other, with forces varying inversely as the square of the distance; because it is assumed as a characteristic property of such media, that the forces of restitution act always in the direction of displacement.

11. *To find the force of restitution, when a particle is slightly disturbed from its position of equilibrium.*

Let F', G', H' be the resolved parts of the force of restitution parallel to the co-ordinate axis upon the particle at P; then F' is the same function of $f+x$, $g+y$, $h+z$, that F is of f, g, h, and therefore

$$F' = F + d_f F \,.\, x + d_g F \,.\, y + d_h F \,.\, z + \dots$$

$$\text{or, } F' = d_f V + d_f^2 V \,.\, x + d_f d_g V \,.\, y + d_f d_h V \,.\, z + \dots$$

$$= d_f^2 V \,.\, x + \text{terms involving } x^2,\ y^2,\ z^2,\ xy, \text{ \&c.}$$

because $d_f V = 0$, $d_f d_g V = 0$, $d_f d_h V = 0$. (Art. 4).

Similarly, $G' = d_g^2 V \,.\, y + \dots$

and $H' = d_h^2 V \,.\, z + \dots$

Hence, if the system consisted of fixed particles, the particle P only being moveable, the equations for P's motion would be

$$\left.\begin{aligned} d_t^2 x &= d_f^2 V \,.\, x \\ d_t^2 y &= d_g^2 V \,.\, y \\ d_t^2 z &= d_h^2 V \,.\, z \end{aligned}\right\} \text{ very nearly.}$$

It is remarkable, that $\dfrac{d_t^2 x}{x} + \dfrac{d_t^2 y}{y} + \dfrac{d_t^2 z}{z} = 0.$

12. From this investigation it appears, that the force of restitution parallel to any one co-ordinate axis depends only upon its displacement

parallel to the same axis. We may therefore consider the effect of each component of the displacement separately.

It appears from the equations just obtained, that $-d_f^2V$, $-d_g^2V$, $-d_h^2V$ are the *absolute* forces of *restitution*.

Since one at least of the quantities d_f^2V, d_g^2V, d_h^2V is negative, and one at least positive, there will be at least one principal axis parallel to which a disturbed particle can vibrate, and at least one parallel to which a disturbed particle *cannot* vibrate. Suppose for instance, that d_f^2V is positive and d_h^2V negative, then the first equation $d_t^2x = d_f^2V.x$ takes the form

$$d_t^2x = \alpha^2x,$$

the integral of which is

$$x = C'e^{\alpha t} + C''e^{-\alpha t};$$

a result which shews that x must increase continually with t. The motion in this direction will therefore be one of *translation*.

But for that part of the displacement which is parallel to the axis of z, the equation of motion is

$$d_t^2z = -\gamma^2z.$$

The integral of which is

$$z = A\cos(\gamma t + B),$$

which denotes vibration.

13. If the constitution, or arrangement of the particles, of the medium is such that d_g^2V is positive, the motion parallel to y will be one of translation; and consequently there will only be one line in which a particle can be displaced, so that its motion may be vibratory.

14. If the constitution of the medium be such that d_g^2V is negative, the motion parallel to y will be vibratory; and therefore if the particle be displaced in any direction in the plane yz, it will continue to vibrate in that plane, describing an elliptic orbit.

15. It appears then that at the most, the equilibrium can only be stable in one *plane;* and that the medium may be so constituted that the equilibrium shall be stable only in one *line.* The character of instability, which in the preceding articles we have shewn necessarily attaches to a medium constituted of particles placed at finite intervals, and *attracting* each other with forces varying as $\frac{1}{D^2}$, cannot be removed by supposing the particles to *repel* each other with forces varying according to the same law. The equation $d_f^2 V + d_g^2 V + d_h^2 V = 0$, from which the instability arises, holds equally for attraction and repulsion.

It may be observed also that the instability cannot be removed by *arrangement;* for though the values of $d_f^2 V$, $d_g^2 V$, $d_h^2 V$ depend upon the arrangement of the particles, the fact that one at least must be positive and one negative depends only upon the equation $d_f^2 V + d_g^2 V + d_h^2 V = 0$, which is true for every arrangement. And consequently, whether the particles be arranged in cubical forms, or in any other manner, there will always exist a direction of instability.

It is therefore certain, that the medium in which luminiferous waves are transmitted to our eyes is not constituted of such particles. The coincidence of numerical results, derived from the hypothesis of a medium of such particles, with experiment, only shews that numerical results are no certain test of theory, when limited to a few cases only.

16. It has been noticed, that the instability of a system depends upon the equation $d_f^2 V + d_g^2 V + d_h^2 V = 0$. With the ordinary law of attraction it always holds good. If, however, the force of molecular attraction be assumed to vary as $\frac{1}{D^n}$, and we write

$$V \text{ for } \Sigma \frac{\left(\frac{m}{r^{n-1}}\right)}{n-1},$$

we shall find

$$d_f^2 V + d_g^2 V + d_h^2 V = (n-2)\, \Sigma \left(\frac{m}{r^{n+1}}\right) \ldots\ldots (1).$$

By an investigation precisely similar to that in Art. 11, we find

$$\left.\begin{aligned} F' &= d_f^2 V . x + \ldots \\ G' &= d_g^2 V . y + \ldots \\ H' &= d_h^2 V . z + \ldots \end{aligned}\right\} \ldots\ldots (2).$$

Now, since $\Sigma\left(\frac{m}{r^{n+1}}\right)$ is necessarily positive, one at least of the quantities, $d_f^2 V$, $d_g^2 V$, $d_h^2 V$, in equation (1) is necessarily positive for all values of n equal to or greater than 2; and consequently, one at least of the equations (2) must necessarily denote *translation*. And this is true whatever be the *arrangement* of the particles.

But when n is less than 2 the right-hand member of (1) is negative, in which case it is possible that all the equations (2) may denote *vibration*.

Hence, if the luminiferous ether consist of detached particles which *attract* each other with forces varying as $\frac{1}{D^n}$, n must be less than 2.

In a similar manner it may be shewn, that if the ether consist of repulsive particles, n must be greater than 2.

It must be remarked, however, that although these conditions with regard to the value of n should be satisfied by the law of attraction of the particles, yet their arrangement must be such as shall make $d_f^2 V$, $d_g^2 V$, $d_h^2 V$ all negative for every particle in the system, otherwise it will be unstable and incapable of transmitting light.

17. If the medium be of the kind denominated uncrystallized, the vibration of a particle in any direction must be completed in the same time, in which case the arrangement must be such as simultaneously to satisfy the equations,

$$d_f^2 V = d_g^2 V = d_h^2 V = \frac{n-2}{3}\ \Sigma\left(\frac{m}{r^{n+1}}\right),$$

n being less than 2.

We shall arrive at the same result if we consider an uncrystallized medium to be such that the force of restitution acts always in the line of displacement; for in this case the parametric surface, the general equation of which is

$$2(n-1)(C-C') = d_f^2 V.x^2 + d_g^2 V.y^2 + d_h^2 V.z^2 \ldots\ldots (\text{Art. } 3).$$

must be spherical; which requires that

$$d_f^2 V = d_g^2 V = d_h^2 V.$$

18. It can be easily shewn that n must be greater than unity.

For the number of particles at the distance r from the attracted particle is proportional to r^2, and therefore

$$\frac{n-2}{3}\Sigma\frac{m}{r^{n+1}} \propto \Sigma\frac{r^2}{r^{n+1}},$$

$$\propto \Sigma\frac{1}{r^{n-1}};$$

hence, unless n be greater than unity, the effect of the more distant parts of the medium upon the value of $\frac{n-2}{3}\Sigma\frac{m}{r^{n+1}}$ will be greater than the effect of the adjacent particles. Now the time of vibration of a particle depends on the value of $d_f^2 V$, or $\frac{n-2}{3}.\Sigma\frac{m}{r^{n+1}}$; and therefore unless n be greater than unity, the parts of the medium which are more remote will exert a greater influence upon the time of vibration than those exert which are near. Now, Optical phenomena seem to indicate that the adjacent particles exercise most influence; and therefore n must be greater than 1.

19. It is probably not conformable to the simplicity of Nature, that n should be fractional; we have shewn that it must be greater than 1 and cannot be equal to 2, consequently n is greater than 2.

This result is important, as we are enabled to infer from it immediately, by the aid of (16), that

If the ethereal medium consist of detached particles, the action of which on each other is proportional to a power of the distance, that power must be greater than 2, and the force must be repulsive.

I have pleasure in remarking, that this result so far as it goes, coincides exactly with that which M. Cauchy has obtained in his "Mémoire sur la dispersion de la lumière," page 191, where from his investigations he infers respecting the mutual action of two molecules of ether, "que, *dans le voisinage du contact, cette action soit répulsive et réciproquement proportionelle au bi-carré de la distance.*"

20. If the particles of ether exert a repulsive action upon each other, as we have just shewn must be the case, they will naturally endeavour to disperse themselves through all space, and form a medium coextensive with the boundaries of the universe. Here then a formidable difficulty presents itself to our notice. If the medium be of finite dimensions it must be enclosed in an envelope, capable of restraining the expansive energy of the whole mass of particles. The more extensive the medium the greater must be the strength of the envelope. Is it probable that the constitution of the Universe is such as to require that the whole should be enclosed in a huge vessel of inconceivable strength? This objection would in my opinion be fatal to the hypothesis of a system of detached particles, were it not for the following considerations.

Upon examining the preceding articles, it will be seen that the luminiferous ether must be such that $d_f^2 V$, $d_g^2 V$, $d_h^2 V$ are all equal and negative. Now the properties of these quantities will not be in the least affected, if we assume that the particles exert *attractive* forces as well as repulsive forces, providing the attractive forces are proportional to $\frac{1}{D^2}$. For let us suppose that

$$V = \Sigma\left(\frac{\mu}{r}\right) - \Sigma\left\{\frac{\left(\frac{m}{r^{n-1}}\right)}{n-1}\right\},$$

where μ and m are respectively the *attractive* and *repulsive* forces exerted by the same particle at the distance unity.

Then as in (16), we have

$$d_f^2 V + d_g^2 V + d_h^2 V = -(n-2)\,\Sigma\left(\frac{m}{r^{n+1}}\right);$$

$$\therefore\ d_f^2 V = d_g^2 V = d_h^2 V = -\frac{n-2}{3}\,\Sigma\left(\frac{m}{r^{n+1}}\right);$$

equations which do not contain the quantity μ.

I think it therefore not improbable, that *each particle of the luminiferous ether exerts two forces, one attractive and varying reciprocally as the square of the distance; and the other repulsive and varying inversely in a higher ratio than the square;* at any rate this supposition does away with the necessity of the envelope mentioned at the beginning of this article.

21. Let us now generalize the problem, and inquire for what laws of molecular force vibration is possible in the particles of ether.

Let ϕr be the law of molecular force; and assume $V = -\Sigma(m \int_r \phi r)$;

$$\therefore\ d_f^2 V + d_g^2 V + d_h^2 V = -\Sigma\left\{m\left(\frac{2\phi r}{r} + \phi' r\right)\right\},$$

$\phi' r$ for brevity denoting $d_r \phi r$.

Now one condition to be fulfilled is, that $d_f^2 V + d_g^2 V + d_h^2 V =$ a negative quantity, and consequently the law of force must be such that

$$\frac{2\phi r}{r} + \phi' r = \text{a positive quantity};$$

for all values of r from $r =$ the distance between two neighbouring particles, to $r = \infty$; let ψr be any function of r which is positive between these limits, then

$$\frac{2\phi r}{r} + \phi' r = \psi r;$$

$$\therefore\ 2r\phi r + r^2 \phi' r = r^2 \psi r,$$

$$\therefore\ r^2 \phi r = C + \int_r (r^2 \psi r),$$

$$\therefore\ \phi r = \frac{C}{r^2} + \frac{1}{r^2}\cdot \int_r (r^2 \psi r).$$

This formula contains every possible law of force: the first term shews the propriety of what we have done in the last article, and further proves, that an attractive molecular force varying inversely as the square of the distance is the only force which possesses the properties requisite for removing the difficulty there stated; or that at any rate it is the simplest and best adapted for that purpose.

Further; for a reason analogous to that assigned in (18),

$$r^2\left(\frac{2\phi r}{r}+\phi' r\right), \text{ or } r^2\psi r$$

must be a function of r, which decreases as r increases, and vanishes when r is infinite. Hence, if $\chi(r)$ be any function of r which is positive between the least and greatest limits of r for the whole medium, and which decreases as r increases and vanishes when r is infinite, then

$$\phi(r)=\frac{C}{r^2}+\frac{1}{r^2}\int_r \chi(r).$$

Every possible law of force is included in this formula; but the converse is not necessarily true, viz. that every law of force included in this formula is possible.

There may be other conditions to be satisfied, either as to the form of the arrangement of the particles, or as to their distance from each other, or as to the possibility of the medium existing in a state of finite extension, or as to other circumstances unknown to us at present which may perhaps exclude all the forms but one; which one would in that case be the actual law in the luminiferous ether. Or there may be peculiarities in the vibrations which constitute the waves of light (such as their *transversality*) which will hereafter enable us to determine the required law of mutual action of the particles.

22. Whatever be the law of molecular force of the luminiferous ether, each particle is placed in such a position when in equilibrium, that the value of V for that particle is a maximum.

Let us employ the notation of (21): then $V=-\Sigma(m\int_r \phi r)$, and

$$d_f^2 V + d_g^2 V + d_h^2 V = -\Sigma\left\{m\left(\frac{2\phi r}{r} + \phi' r\right)\right\}$$
$$= -\Sigma(m\psi r),$$

and every one of the quantities $d_f^2 V$, $d_g^2 V$, $d_h^2 V$ is negative, whether the particle of ether (the state of which we are investigating) be within a crystallized body, or in vacuo, or in an uncrystallized body.

In order that V may be a maximum, we must have fulfilled the following conditions, viz.

$$d_f V = 0, \quad d_g V = 0, \quad d_h V = 0 \quad \ldots\ldots (1),$$

$$d_f^2 V, \quad d_g^2 V, \quad d_h^2 V \text{ all negative} \quad \ldots\ldots (2),$$

and

$$\left.\begin{array}{l} d_f^2 V . d_g^2 V > (d_f d_g V)^2 \\ d_g^2 V . d_h^2 V > (d_g d_h V)^2 \\ d_f^2 V . d_h^2 V > (d_f d_h V)^2 \end{array}\right\} \quad \ldots\ldots (3).$$

The three conditions marked (1) are fulfilled, because the particle is in equilibrium by hypothesis; we have shewn above that the three conditions (2) are fulfilled, otherwise the medium could not be luminiferous, *i. e.* its particles could not vibrate in *any* direction; and the last three conditions marked (3) are fulfilled, because the directions of the co-ordinate axes have been taken, such that $d_f d_g V = 0$, $d_g d_h V = 0$, and $d_f d_h V = 0$. Consequently V is a maximum.

S. EARNSHAW.

VI. *Supplement to a Memoir on the Reflexion and Refraction of Light.* *By* G. Green, Esq. B.A. *of Caius College.*

[Read *May* 6, 1839.]

In a paper which the Society did me the honour to publish some time ago, I endeavoured to determine the laws of Reflexion and Refraction of a plane wave at the surface of separation of two elastic media, supposing this surface perfectly plane, and both media to terminate there abruptly: neglecting also all extraneous forces, whether due to the action of the solid particles of transparent bodies on the elastic medium, which is supposed to pervade their interstices, or to extraneous pressures. I am inclined to think that in the case of non-crystallized bodies the latter cause would not alter the *form* of our results in the slightest degree; and possibly there would be some difficulty in submitting the effects of the former to calculation. Moreover, should the radius of the sphere of sensible action of the molecular forces bear any finite ratio to λ, the length of a wave of light, as some philosophers have supposed, in order to explain the phenomena of dispersion, instead of an abrupt termination of our two media we should have a continuous though rapid change of state of the ethereal medium in the immediate vicinity of their surface of separation. And I have here endeavoured to shew, by probable reasoning, that the effect of such a change would be to diminish greatly the quantity of light reflected at the polarizing angle, even for highly refracting substances: supposing the light polarized perpendicular to the plane of incidence. The same reasoning would go to prove that in this case the quantity of the reflected light would depend greatly on minute changes in the state of the reflecting surface. I have on the present occasion merely noticed, but not insisted upon, these inferences, feeling persuaded that in researches like the present, little confidence is due to such consequences as are not supported by a rigorous analysis.

The principal object of this supplement has been to put the equations due to the surface of junction of two media, and belonging to light polarized perpendicular to the plane of incidence, under a more simple form. The resulting expressions have here been made to depend on those before given in our paper on Sound, and thus the determination of the intensities of the reflected and refracted waves becomes in every case a matter of extreme facility. As an example of the use of the new formulæ, the intensities of the refracted waves have been determined for both kinds of light: the consideration of which waves had inadvertently been omitted in a former communication.

Perhaps I may be permitted on the present occasion to state, that though I feel great confidence in the truth of the fundamental principle on which our reasonings concerning the vibrations of elastic media have been based, the same degree of confidence is by no means extended to those adventitious suppositions which have been introduced for the sake of simplifying the analysis.

Let us here resume the equations of the paper before mentioned, namely,

$$(17)\qquad \left.\begin{aligned} \frac{d\phi}{dx} + \frac{d\psi}{dy} &= \frac{d\phi_{\prime}}{dx} + \frac{d\psi_{\prime}}{dy} \\ \frac{d\phi}{dy} - \frac{d\psi}{dx} &= \frac{d\phi_{\prime}}{dy} - \frac{d\psi_{\prime}}{dx} \\ \frac{d^2\phi}{g^2 dt^2} &= \frac{d^2\phi_{\prime}}{g_{\prime}^2 dt^2} \\ \frac{d^2\psi}{\gamma^2 dt^2} &= \frac{d^2\psi_{\prime}}{\gamma_{\prime}^2 dt^2} \end{aligned}\right\} \text{(when } x = 0\text{)}.$$

where u and v, the disturbances in the upper medium parallel to the axes x and y, are given by

$$u = \frac{d\phi}{dx} + \frac{d\psi}{dy},$$

$$v = \frac{d\phi}{dy} - \frac{d\psi}{dx};$$

$u_{,}$ and $v_{,}$ the disturbances in the lower medium being expressed by similar formulæ in $\phi_{,}$ and $\psi_{,}$.

The two last equations of (17) give, since

$$\mu = \frac{g}{g_{,}} = \frac{\gamma}{\gamma_{,}},$$

$$\phi' = \mu^2 \phi_{,}', \qquad \psi' = \mu^2 \psi_{,}';$$

ϕ and $\phi_{,}$ being accented for a moment to distinguish between the particular values belonging to the plane (yz) and their more general values

$$\phi = \epsilon^{bx}\phi' \quad \text{and} \quad \phi_{,} = \epsilon^{-bx}\phi_{,}'.$$

The correctness of these values will be evident on referring to the Memoir, formulæ (20), (21), and recollecting that

$$b = a' = a_{,}'.$$

Hence the first equation gives, since $x = 0$,

$$b(\mu^2 + 1)\,\phi_{,}' = \frac{d\psi_{,}}{dy} - \frac{d\psi}{dy} = -(\mu^2 - 1)\frac{d\psi_{,}}{dy};$$

$$\therefore \quad \phi_{,}' = -\frac{\mu^2 - 1}{b(\mu^2 + 1)}\frac{d\psi_{,}}{dy}, \quad \text{and } \phi' = \frac{-\mu^2(\mu^2 - 1)}{b(\mu^2 + 1)}\frac{d\psi_{,}}{dy}.$$

Also the second equation may be written,

$$\frac{d\psi}{dx} - \frac{d\psi_{,}}{dx} = \frac{d\phi'}{dy} - \frac{d\phi_{,}'}{dy} = -\frac{(\mu^2 - 1)^2}{b(\mu^2 + 1)}\frac{d^2\psi_{,}}{dy^2}.$$

And since we may differentiate or integrate the equations (17) relative to any variable except x, we get for the conditions requisite to complete the determination of ψ and $\psi_{,}$,

$$(29)\qquad \left.\begin{aligned} \psi &= \mu^2\psi_{,} \\ \frac{d\psi}{dx} &= \frac{d\psi_{,}}{dx} - \frac{(\mu^2-1)^2}{(\mu^2+1)b}\frac{d^2\psi_{,}}{dx^2} \end{aligned}\right\}\text{(when } x = 0).$$

Or neglecting the term which is insensible except for highly refracting substances,

$$(30)\qquad \left.\begin{aligned} \psi &= \mu^2\psi_{,} \\ \frac{d\psi}{dx} &= \frac{d\psi_{,}}{dx} \end{aligned}\right\}\text{(when } x = 0),$$

* where $\mu = \frac{\gamma}{\gamma_{,}}$ is the index of refraction.

These equations belong to light polarized in a plane perpendicular to that of incidence, and as ϕ and $\phi_{,}$ are insensible at sensible distances from the surface of junction of the two media, we have, except in the immediate vicinity of this surface,

$$(31)\qquad \begin{aligned} u &= \frac{d\psi}{dy} \\ v &= -\frac{d\psi}{dx}. \end{aligned}$$

* Though these equations have been obtained on the supposition that the vibrations of the media follow the law of the cycloidal pendulum, yet as (b) has disappeared, they are equally applicable for all plane waves whatever.

In fact, instead of using the value

$$\psi_{,} = \alpha_{,} \sin(a_{,}x + by + ct),$$

and corresponding values of the other quantities, we might have taken the infinite series

$$\psi_{,} = \Sigma\alpha_{,} \sin n(a_{,}x + by + ct),$$

where α and n may have any series of values at will. But the last expression is the equivalent of an arbitrary function of

$$a_{,}x + by + ct.$$

Or the same equations might have been immediately obtained from (17), without introducing this consideration. The method in the text has been employed for the sake of the intermediate result (29), of which we shall afterwards make use.

When light is polarized in the plane of incidence, the conditions at the surface of junction have been shewn to be

$$(32)\qquad \left.\begin{aligned} w &= w_{,} \\ \frac{dw}{dx} &= \frac{dw_{,}}{dx} \end{aligned}\right\} \text{(when } x = 0\text{)}.$$

Since in these conditions we may differentiate or integrate relative to any of the independent variables except x, we see that the expressions (30) and (32) are reduced to a form equivalent to that marked (A) in our paper on Sound; and the general equations in ψ and w being the same, we may immediately obtain the intensity of the reflected or refracted waves, by merely writing in the simple formulæ contained in that paper,

$\Delta = 1$ and $\Delta_{,} = 1$ for light polarized in the plane of incidence;

or $\Delta = \dfrac{1}{\gamma^2}$ and $\Delta_{,} = \dfrac{1}{\gamma_{,}^2}$ for light polarized perpendicular to the plane of incidence.

As an example, we will here deduce the intensity of the refracted wave for both kinds of light.

Representing, therefore, the parts of w and $w_{,}$ due to the disturbances in the Incident Reflected and Refracted waves by

$$f(ax + by + ct),\quad F(-ax + by + ct),\quad \text{and } f_{,}(a_{,}x + by + ct)$$

respectively, and resuming the first of our expressions (7) in the paper on Sound, viz.—

$$f' = \tfrac{1}{2}\left(\frac{\Delta_{,}}{\Delta} + \frac{a_{,}}{a}\right) f_{,}',$$

we get for light polarized in the plane of incidence, where $\Delta = \Delta_{,} = 1$,

$$\frac{f_{,}}{f} = \frac{2}{1 + \dfrac{a_{,}}{a}} = \frac{2}{1 + \dfrac{\cot\theta_{,}}{\cot\theta}} = \frac{2\cos\theta\sin\theta_{,}}{\sin(\theta_{,} + \theta)},$$

which agrees with the value given in Airy's Tracts, p. 356.

For light polarized perpendicular to the plane of incidence, we have $\Delta = \dfrac{1}{\gamma^2}$ and $\Delta_{,} = \dfrac{1}{\gamma_{,}^2}$. If, therefore, we here represent the parts of ψ and $\psi_{,}$ due to the same disturbances by f, F and $f_{,}$, we get

$$\frac{f_{,}'}{f'} = \frac{2}{\dfrac{\gamma^2}{\gamma_{,}^2} + \dfrac{\cot\theta_{,}}{\cot\theta}} = \frac{\sin\theta_{,}\cos\theta}{\sin\theta\cos\theta_{,}} \cdot \frac{2}{\dfrac{\cos\theta\sin\theta}{\cos\theta_{,}\sin\theta_{,}} + 1}.$$

Also, if D be the disturbance of the incident wave in its own plane, and $D_{,}$ the like disturbance in the refracted wave, we have by first equation of (31),

$$D\sin\theta = u = \frac{d\psi}{dy} = bf'(ax + by + ct),$$

$$\text{and } D_{,}\sin\theta_{,} = u_{,} = \frac{d\psi_{,}}{dy} = bf_{,}'(ax + by + ct),$$

retaining in ψ the part due to the incident wave only.

Thus by writing the characteristics merely,

$$\frac{D_{,}}{D} = \frac{\sin\theta}{\sin\theta_{,}}\frac{f_{,}'}{f'} = \frac{\cos\theta}{\cos\theta_{,}} \cdot \frac{2}{\dfrac{\cos\theta\sin\theta}{\cos\theta_{,}\sin\theta_{,}} + 1}$$

$$= \frac{\cos\theta}{\cos\theta_{,}}\left\{1 + \frac{-\dfrac{\cos\theta\sin\theta}{\cos\theta_{,}\sin\theta_{,}} + 1}{\dfrac{\cos\theta\sin\theta}{\cos\theta_{,}\sin\theta_{,}} + 1}\right\} = \frac{\cos\theta}{\cos\theta_{,}}\left\{1 + \frac{\tan(\theta_{,} - \theta)}{\tan(\theta + \theta_{,})}\right\},$$

which agrees with the formula in use. (Vide Airy's Tracts, p. 358).

In our preceding paper, the two media have been supposed to terminate abruptly at their surface of junction, which would not be true of the luminiferous ether, unless the radius of the sphere of sensible action of the molecular forces was exceedingly small compared with λ, the length of a wave of light.

In order, therefore, to form an estimate of the effect which would be produced by a continuous though rapid change of state of the ethereal medium in the immediate vicinity of the surface of junction, we will resume the conditions (29), which belong to light polarized in a plane perpendicular to that of Reflexion, viz.

$$(29) \qquad \psi = \mu^2 \psi_,, \text{ and } \frac{d\psi}{dx} = \frac{d\psi_,}{dx} - \frac{(\mu^2 - 1)^2}{(\mu^2+1)b} \frac{d^2\psi_,}{dx^2} \; (x = 0);$$

and instead of supposing the index of refraction to change suddenly from 0 to μ, we will conceive it to pass through the regular series of gradations,

$$\mu_0, \; \mu_1, \; \mu_2, \; \mu_3 \ldots\ldots\ldots\ldots \mu_n;$$

τ being the common thickness of each of these successive media.

Then it is clear we should have to replace the last system by

$$\mu_0\psi_0 = \mu_1^2\psi_1, \text{ and } \frac{d\psi_0}{dx} = \frac{d\psi_1}{dx} - \frac{(\mu_1^2 - \mu_0^2)^2}{\mu_0^2(\mu_1^2 + \mu_0^2)\,b} \frac{d^2\psi_1}{dx^2} \; (x = 0),$$

$$(33) \qquad \mu_1^2\psi_1 = \mu_2^2\psi_2, \text{ and } \frac{d\psi_1}{dx} = \frac{d\psi_2}{dx} - \frac{(\mu_2^2 - \mu_1^2)^2}{\mu_1^2(\mu_2^2 + \mu_1^2)\,b} \frac{d^2\psi_2}{dx^2} \; (x = \tau),$$

$$\mu_2^2\psi_2 = \mu_3^2\psi_3, \text{ and } \frac{d\psi_2}{dx} = \frac{d\psi_3}{dx} - \frac{(\mu_3^2 - \mu_2^2)^2}{\mu_2^2(\mu_3^2 + \mu_2^2)\,b} \frac{d^2\psi_3}{dx^2} \; (x = 2\tau),$$

....

$$\mu_{n-1}^2\psi_{n-1} = \mu_n^2\psi_n, \text{ and } \frac{d\psi_{n-1}}{dx} = \frac{d\psi_n}{dx} - \frac{(\mu_n^2 - \mu_{n-1}^2)^2}{\mu_{n-1}^2(\mu_n^2 + \mu_{n-1}^2)\,b} \frac{d^2\psi_n}{dx^2} \; \{x = (n-1\,.\,\tau)\}.$$

But it is evident from the form of the equations on the right side of system (33), that the total effect due to the last terms of their second members will be far less when n is great, than that due to the corresponding term in the second equation of system (29)*. If, therefore, we reject these second terms, and conceive the common interval τ so small that the result due to the first terms may not differ very sensibly from that which would be produced by a single refraction, we should have to replace the system (29) by (30), and the intensity of the reflected wave would then agree with the law assigned by Fresnel. In virtue of this law, however highly refracting any substance may be, homogeneous light will always be completely polarized at a certain angle of incidence; and Sir David Brewster states that this is the case with diamond at the proper angle. But the phenomena observed by Professor Airy appear to him entirely inconsistent with this result (Vide *Camb. Phil. Trans.*, Vol. IV. p. 423.); what immediately precedes seems to render it probable that considerable differences in this respect may be due to slight changes in the reflecting surface.

* In fact, in the system (33) each of the last terms will, in consequence of the factors $(\mu_{,}^2 - \mu_0^2)^2$, &c. be quantities of the order $\frac{1}{n^2}$ compared with the last term of (29′), and as their number is only n, their joint effect will be a quantity of the order $\frac{1}{n}$ compared with that of the term just mentioned.

Fig I

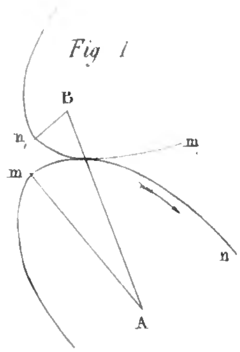


Fig II

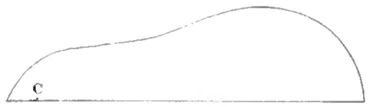


Fig III

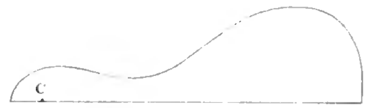


Fig. IV

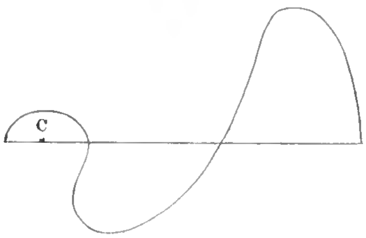


Fig. V

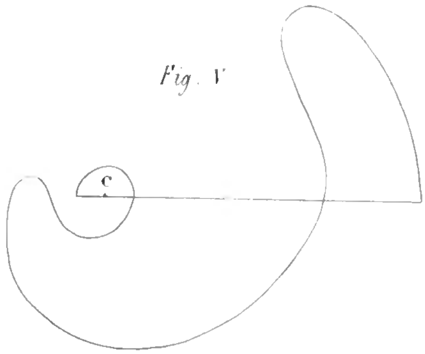
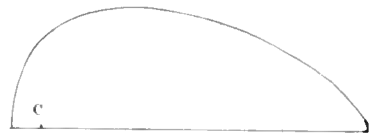


Fig. VI



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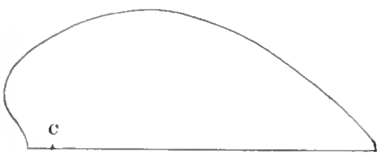

Fig. VIII

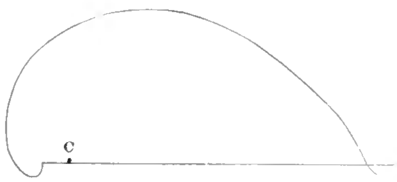

Fig IX

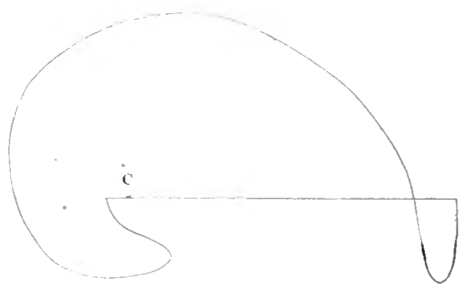

Fig. X

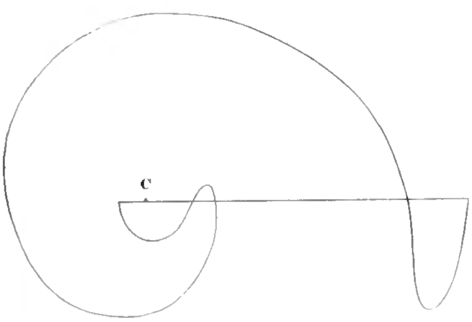

Fig XXV

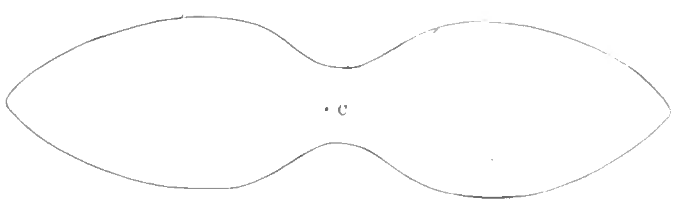

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Fig. XI

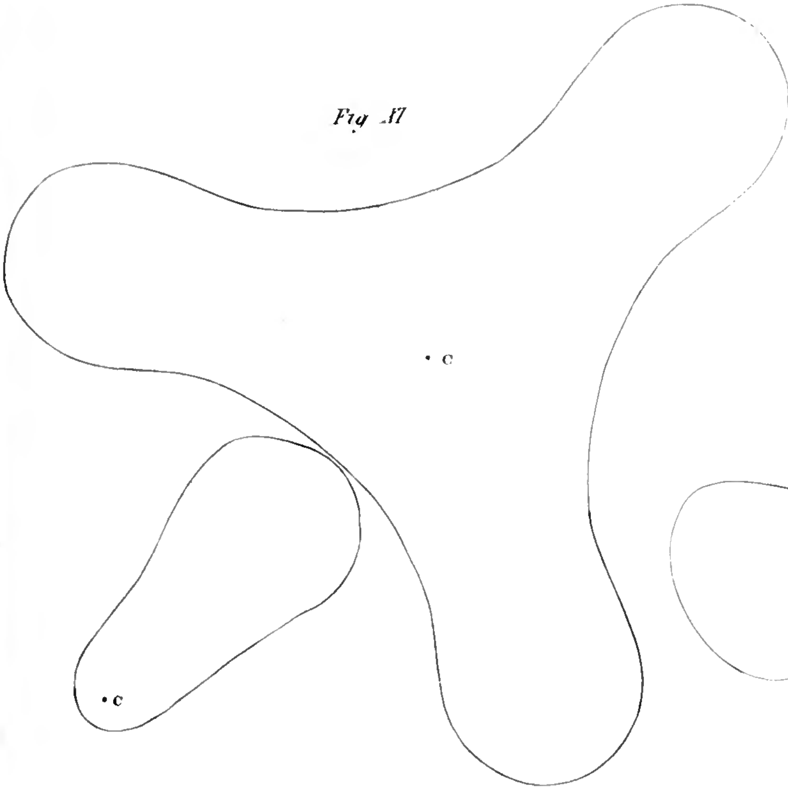


Fig. XII

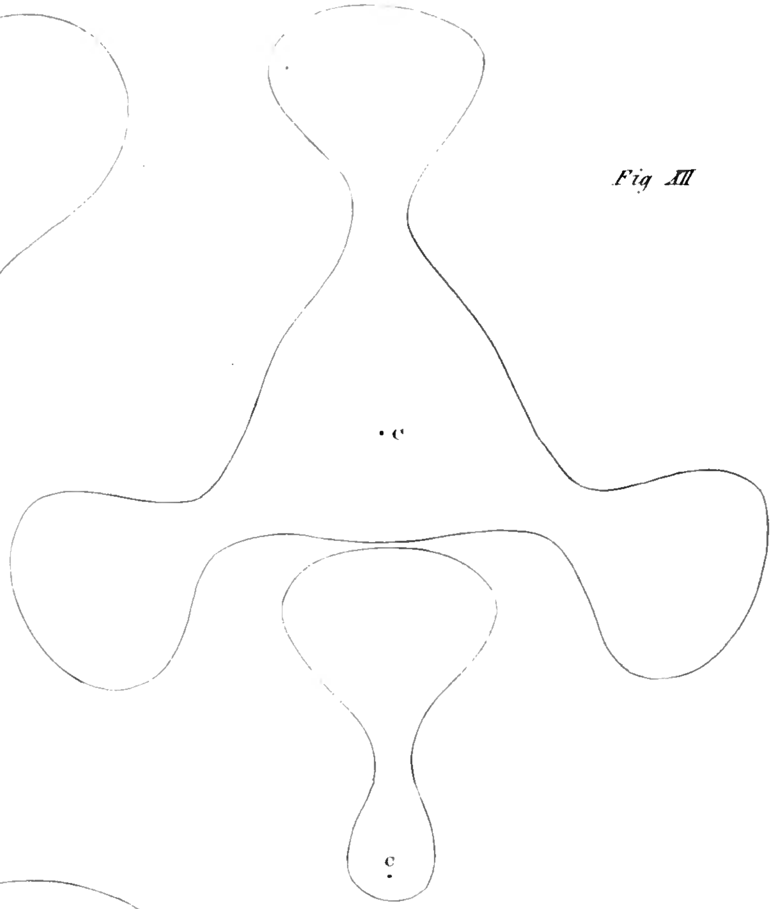


Fig. XIII

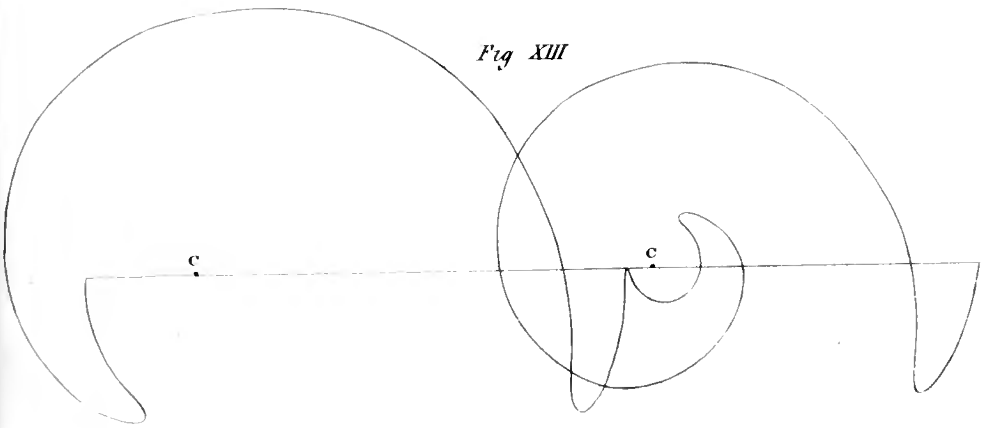


Fig. XXII

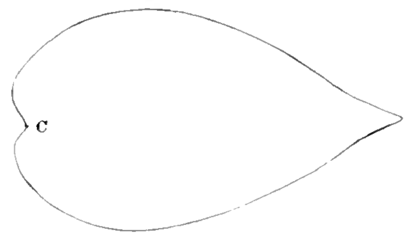


Fig. XIV

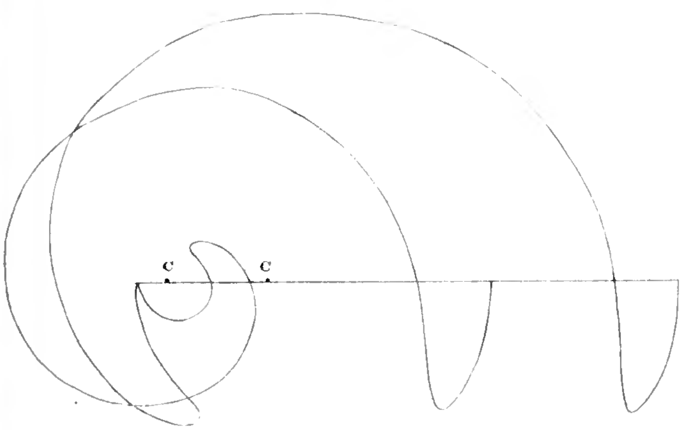
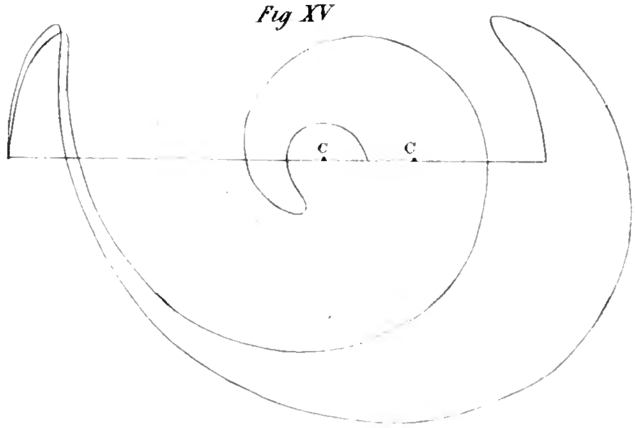


Fig. XV



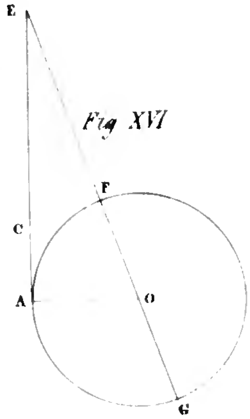


Fig XVI

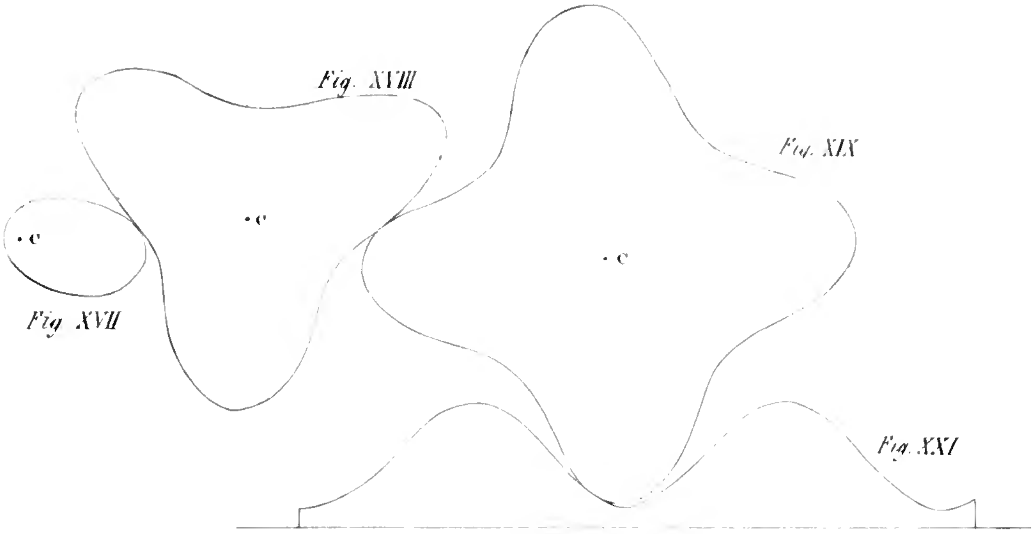


Fig. XVII

Fig. XVIII

Fig. XIX

Fig. XXI

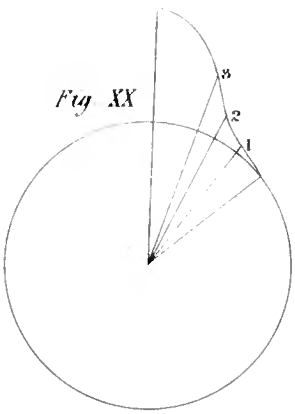


Fig. XX

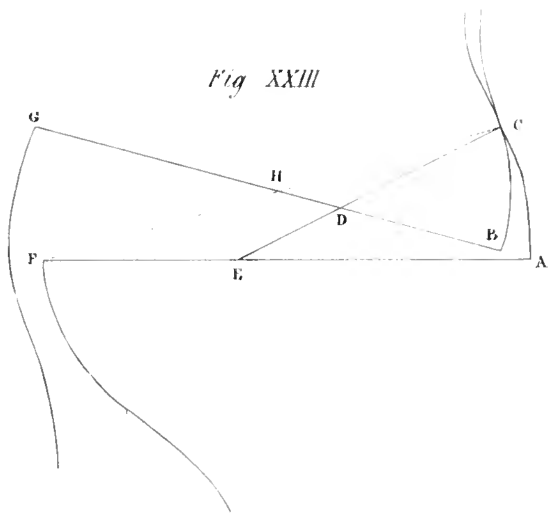


Fig. XXIII

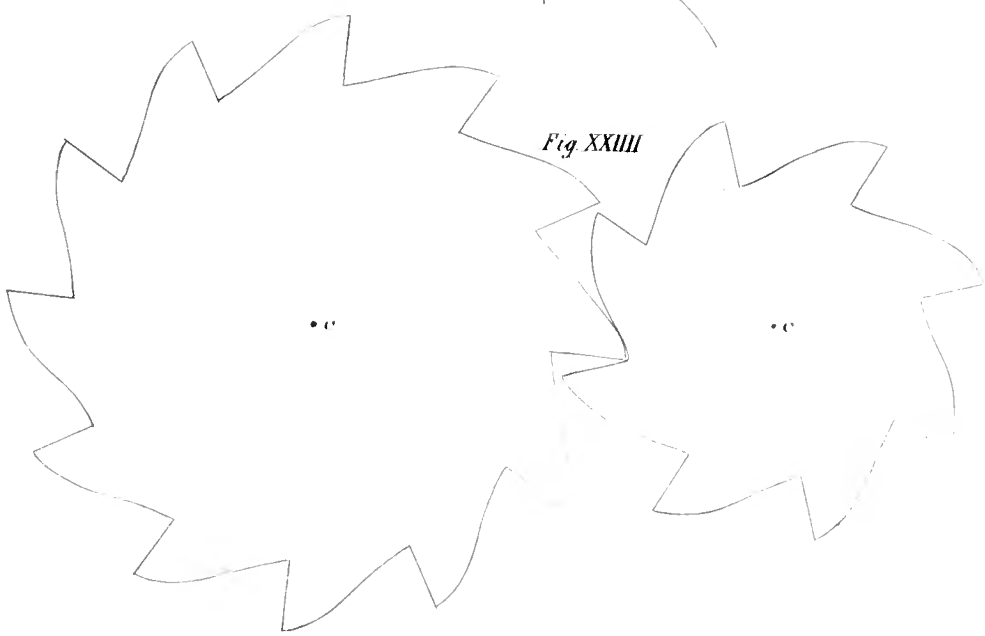


Fig. XXIIII

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MDCCC.XLI.

VII. *On the Propagation of Light in Crystallized Media.* *By* G. GREEN, B.A. *Fellow of Caius College.*

[Read *May* 20, 1839.]

IN a former paper I endeavoured to determine in what way a plane wave would be modified when transmitted from one non-crystallized medium to another; founding the investigation on this principle: In whatever manner the elements of any material system may act upon each other, if all the internal forces be multiplied by the elements of their respective directions, the total sum for any assigned portion of the mass will always be the exact differential of some function. This principle requires a slight limitation, and when the necessary limitation is introduced, appears to possess very great generality. I shall here endeavour to apply the same principle to crystallized bodies, and shall likewise introduce the consideration of the effects of extraneous pressures, which had been omitted in the former communication. Our problem thus becomes very complicated, as the function due to the internal forces, even when there are no extraneous pressures, contains twenty-one coefficients. But with these pressures we are obliged to introduce six additional coefficients; so that without some limitation, it appears quite hopeless thence to deduce any consequences which could have the least chance of a physical application. The absolute necessity of introducing some arbitrary restrictions, and the desire that their number should be as small as possible, induced me to examine how far our function would be limited by confining ourselves to the consideration of those media only in which the directions of the transverse vibrations shall always be *accurately* in the front of the wave. This fundamental principle of Fresnel's Theory gives fourteen relations between the twenty-one constants originally entering into our function; and it seems worthy of remark, that when there are no extraneous pressures, the directions of polarization and the wave-velocities given by our theory, when thus limited, are identical with those assigned by Fresnel's general construction for biaxal crystals; provided we suppose the actual direction of disturbance in the particles of the medium is *parallel* to the plane of polarization, agreeably to the supposition first advanced by M. Cauchy.

If we admit the existence of extraneous pressures, it will be necessary, in addition to the single restriction before noticed, to suppose that for three plane waves parallel to three orthogonal sections of our medium, and which may be denominated principal sections, the wave-velocities shall be the same for any two of the three waves whose fronts are parallel to these sections, provided the direction of the corresponding disturbances are parallel to the line of their intersection. With this additional supposition, the directions of the actual disturbances by which any plane wave will propagate itself without subdivision, and the wave-velocities agree exactly with those given by Fresnel, supposing, with him, that these directions are *perpendicular* to the plane of polarization. The last, or Fresnel's hypothesis, was adopted in our former paper. But as that paper relates merely to the intensities of the waves reflected and refracted at the surface of separation of two media, and as these intensities may depend upon physical circumstances, the consideration of which was not introduced into our former investigations, it seems right, in the present paper, considering the actual situation of the theory of light, when the partial differential equations on which the determination of the motion of the luminiferous ether depends are yet to discover, to state fairly the results of both hypotheses.

It is hoped the analysis employed on the present occasion will be found sufficiently simple, as a method has here been given of passing immediately and without calculation from the function due to the internal forces of our medium to the equation of an ellipsoidal surface, of which the semi-axes represent in magnitude the reciprocals of the three wave-velocities, and in direction the directions of the three corresponding disturbances by which a wave can propagate itself in our medium without subdivision. This surface, which may be properly styled the ellipsoid of elasticity, must not be confounded with the one whose section by a plane parallel to the wave's front gives the reciprocals of the wave-velocities, and the corresponding directions of polarization. The two surfaces have only this section in common, and a very simple application of our theory would shew that no force perpendicular to the wave's front is rejected, as in the ordinary one, but that the force in question is absolutely null.

Let us conceive a system composed of an immense number of particles mutually acting on each other, and moreover subjected to the influence of extraneous pressures. Then if x, y, z are the co-ordinates of any particle of this system in its primitive state, (that of equilibrium under pressure for example,) the co-ordinates of the same particle at the end of the time t will become x', y', z', where $x'\ y'\ z'$ are functions of $x\ y\ z$ and t. If now we consider an element of this medium, of which the primitive form is that of a rectangular parallelopiped, whose sides are dx, dy, dz, this element in its new state will assume the form of an oblique-angled parallelopiped, the lengths of the three edges being $(dx'), (dy'), (dz')$, these edges being composed of the same particles which formed the three edges dx, dy, dz in the primitive state of the element. Then will

$$\left.\begin{aligned}(dx')^2 &= \left\{\left(\frac{dx'}{dx}\right)^2 + \left(\frac{dy'}{dx}\right)^2 + \left(\frac{dz'}{dx}\right)^2\right\} dx^2 = a^2dx^2\\(dy')^2 &= \left\{\left(\frac{dx'}{dy}\right)^2 + \left(\frac{dy'}{dy}\right)^2 + \left(\frac{dz'}{dy}\right)^2\right\} dy^2 = b^2dy^2\\(dz')^2 &= \left\{\left(\frac{dx'}{dz}\right)^2 + \left(\frac{dy'}{dz}\right)^2 + \left(\frac{dz'}{dz}\right)^2\right\} dz^2 = c^2dz^2\end{aligned}\right\} \text{ suppose.}$$

Again, let

$$\alpha = \cos <\frac{(dy')}{(dz')} = \frac{\frac{dx'}{dy}\frac{dx'}{dz} + \frac{dy'}{dy}\frac{dy'}{dz} + \frac{dz'}{dy}\frac{dz'}{dz}}{\sqrt{\left\{\left(\frac{dx'}{dy}\right)^2 + \left(\frac{dy'}{dy}\right)^2 + \left(\frac{dz'}{dy}\right)^2\right\}\left\{\left(\frac{dx'}{dz}\right)^2 + \left(\frac{dy'}{dz}\right)^2 + \left(\frac{dz'}{dz}\right)^2\right\}}},$$

$$\beta = \cos < \frac{(dx')}{(dz')} = \frac{\frac{dx'}{dx}\frac{dx'}{dz} + \frac{dy'}{dx}\frac{dy'}{dz} + \frac{dz'}{dx}\frac{dz'}{dz}}{\sqrt{\left\{\left(\frac{dx'}{dx}\right)^2 + \left(\frac{dy'}{dx}\right)^2 + \left(\frac{dz'}{dx}\right)^2\right\}\left\{\left(\frac{dx'}{dz}\right)^2 + \left(\frac{dy'}{dz}\right)^2 + \left(\frac{dz'}{dz}\right)^2\right\}}},$$

$$\gamma = \cos < \frac{(dx')}{(dy')} = \frac{\frac{dx'}{dx}\frac{dx'}{dy} + \frac{dy'}{dx}\frac{dy'}{dy} + \frac{dz'}{dx}\frac{dz'}{dy}}{\sqrt{\left\{\left(\frac{dx'}{dx}\right)^2 + \left(\frac{dy'}{dx}\right)^2 + \left(\frac{dz'}{dx}\right)^2\right\}\left\{\left(\frac{dx'}{dy}\right)^2 + \left(\frac{dy'}{dy}\right)^2 + \left(\frac{dz'}{dy}\right)^2\right\}}},$$

or we may write

$$\alpha' = bc\alpha = \frac{dx'}{dy}\frac{dx'}{dz} + \frac{dy'}{dy}\frac{dy'}{dz} + \frac{dz'}{dy}\frac{dz'}{dz},$$

$$\beta' = ac\beta = \frac{dx'}{dx}\frac{dx'}{dz} + \frac{dy'}{dx}\frac{dy'}{dz} + \frac{dz'}{dx}\frac{dz'}{dz},$$

$$\gamma' = ab\gamma = \frac{dx'}{dx}\frac{dx'}{dy} + \frac{dy'}{dx}\frac{dy'}{dy} + \frac{dz'}{dx}\frac{dz'}{dy}.$$

Suppose now, as in a former paper, that $\phi\, dx\, dy\, dz$ is the function due to the mutual actions of the particles which compose the element whose primitive volume $= dx\, dy\, dz$. Since ϕ must remain the same, when the sides (dx') (dy') (dz') and the cosines α, β, γ of the angles of the elementary oblique-angled parallelopiped remain unchanged, its most general form must be

$$\phi = \text{Function } (a, b, c, \alpha, \beta, \gamma)$$

or since a b and c are necessarily positive, also

$$\alpha' = bc\alpha, \; \beta' = ac\beta, \text{ and } \gamma' = ab\gamma,$$

we may write

$$\phi = f\,(a^2, b^2, c^2\; \alpha', \beta', \gamma'). \qquad (1.)$$

This expression is the equivalent of the one immediately preceding, and is here adopted for the sake of introducing greater symmetry into our formulæ.

We will in the first place suppose that ϕ is symmetrical with regard to three planes at right angles to each other, which we shall take as the co-ordinate planes. The condition of symmetry with respect to the plane (yz), will require ϕ to remain unchanged, when we change

$$\left.\begin{matrix} x \\ x' \end{matrix}\right\} \text{into} \left\{\begin{matrix} -x \\ -x'. \end{matrix}\right.$$

But thus a^2, b^2, c^2 and α' evidently remain unaltered; moreover,

$$\left.\begin{matrix} \beta' \\ \gamma' \end{matrix}\right\} \text{become} \left\{\begin{matrix} -\beta' \\ -\gamma'. \end{matrix}\right.$$

Hence we get

$$\phi = f\,(a^2,\ b^2,\ c^2,\ \alpha',\ \beta'^2,\ \gamma'^2).$$

Applying like reasoning to the other co-ordinate planes, we see that the ultimate result will be

$$\phi = f\,(a^2,\ b^2,\ c^2,\ \alpha'^2,\ \beta'^2,\ \gamma'^2). \qquad (2.)$$

The foregoing values are perfectly general, whatever the disturbance may be; but if we consider this disturbance as very small, we may make

$$\begin{aligned} x' &= x + u, \\ y' &= y + v, \\ z' &= z + w, \end{aligned}$$

u, v and w being very small functions of x, y, z and t of the first order. Then by substitution we get

$$\left.\begin{aligned} a^2 &= 1 + 2\frac{du}{dx} + \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2 = 1 + s_1 \\ b^2 &= 1 + 2\frac{dv}{dy} + \left(\frac{du}{dy}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dy}\right)^2 = 1 + s_2 \\ c^2 &= 1 + 2\frac{dw}{dz} + \left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2 + \left(\frac{dw}{dz}\right) = 1 + s_3 \end{aligned}\right\} \text{suppose.}$$

(3.)

$$\alpha' = \frac{dv}{dz} + \frac{dw}{dy} + \frac{du}{dy}\,\frac{du}{dz} + \frac{dv}{dy}\,\frac{dv}{dz} + \frac{dw}{dy}\cdot\frac{dw}{dz}$$

$$\beta' = \frac{du}{dz} + \frac{dw}{dx} + \frac{du}{dx}\,\frac{du}{dz} + \frac{dv}{dx}\,\frac{dv}{dz} + \frac{dw}{dx}\,\frac{dw}{dz}$$

$$\gamma' = \frac{du}{dy} + \frac{dv}{dx} + \frac{du}{dx}\,\frac{du}{dy} + \frac{dv}{dx}\,\frac{dv}{dy} + \frac{dw}{dx}\,\frac{dw}{dy}$$

we thus see that s_1, s_2, s_3, α', β', γ', are very small quantities of the first order, and that the general formula (1) by substituting the preceding values would take the form

$$\phi = \text{Function } (s_1,\ s_2,\ s_3,\ \alpha',\ \beta',\ \gamma')$$

which may be expanded in a very convergent series of the form

$$\phi = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \&c.$$

ϕ_0 ϕ_1 ϕ_2 &c. being homogeneous functions of s_1, s_2, s_3, α', β', γ' of the degrees 0 . 1 . 2 . 3 &c. each of which is very great compared with the next following one.

But ϕ_0, being constant, if $\rho =$ the primitive density of the element, the general formula of Dynamics will give

$$\iiint \rho\, dx\, dy\, dz \left\{\frac{d^2u}{dt^2}\,\delta u + \frac{d^2v}{dt^2}\,\delta v + \frac{d^2w}{dt^2}\,\delta w\right\} = \iiint dx\, dy\, dz \left(\delta\phi_1 + \delta\phi_2 + \&c.\right)$$

If there were no extraneous pressures, the supposition that the primitive state was one of equilibrium would require $\phi_1 = 0$, as was observed in a former paper; but this is not the case if we introduce the consideration of extraneous pressures. However, as in the first case, the terms ϕ_3 ϕ_4, &c., will be insensible, and the preceding formula may be written

$$\iiint \rho\, dx\, dy\, dz \left\{\frac{d^2u}{dt^2}\,\delta u + \frac{d^2v}{dt^2}\,\delta v + \frac{d^2w}{dt^2}\,\delta w\right\} = \iiint dx\, dy\, dz \left(\delta\phi_1 + \delta\phi_2\right)$$

Supposing ρ the primitive density constant, the most general form of ϕ_1 will be

$$\phi_1 = -\tfrac{1}{2}(As_1 + Bs_2 + Cs_3 + 2D\alpha' + 2E\beta' + 2F\gamma'),$$

$ABCDE$ and F being constant quantities.

In like manner the most general form of ϕ_2 will contain twenty-one coefficients. But if we first employ the more particular value, (2) we shall get

$$-2\phi_1 = As_1 + Bs_2 + Cs_3$$

$$-2\phi_2 = Gs_1^2 + Hs_2^2 + Is_3^2 + 2Ps_2s_3 + 2Qs_1s_3 + 2Rs_1s_2 + L\alpha'^2 + M\beta'^2 + N\gamma'^2.$$

Or by substituting for s_1, s_2, s_3, α', β', γ' their values, given by system (3), continuing to neglect quantities of the third order, we get

$$\begin{aligned}
-2\phi &= -2\phi_1 - 2\phi_2 \\
&= 2A\frac{du}{dx} + 2B\frac{dv}{dy} + 2C\frac{dw}{dz} \\
&\quad + A\left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right\} \\
&\quad + B\left\{\left(\frac{du}{dy}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dy}\right)^2\right\} \qquad (4.) \\
&\quad + C\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2 + \left(\frac{dw}{dz}\right)^2\right\} \\
&\quad + G\left(\frac{du}{dx}\right)^2 + H\left(\frac{dv}{dy}\right)^2 + I\left(\frac{dw}{dz}\right)^2 + 2P\frac{dv}{dy}\frac{dw}{dz} + 2Q\frac{du}{dx}\frac{dw}{dz} \\
&\quad + 2R\frac{du}{dx}\frac{dv}{dy} + L\left(\frac{dv}{dz} + \frac{dw}{dy}\right)^2 + M\left(\frac{du}{dz} + \frac{dw}{dx}\right)^2 + N\left(\frac{du}{dy} + \frac{dv}{dx}\right)^2.
\end{aligned}$$

Having thus the form of the function due to the internal actions of the particles, we have merely to substitute it in the general formula of Dynamics, and to effect the integrations by parts, agreeably to the method of Lagrange. Thus,

$$\iiint dx\, dy\, dz\, \delta\phi =$$

$$-\iint dy\, dz \left\{ A\,\delta u + A\left(\frac{du}{dx}\delta u + \frac{dv}{dx}\delta v + \frac{dw}{dx}\delta w\right)\right.$$

$$\left. + \left(G\frac{du}{dx} + R\frac{dv}{dy} + Q\frac{dw}{dz}\right)\delta u + M\left(\frac{du}{dz} + \frac{dw}{dx}\right)\delta w + N\left(\frac{du}{dy} + \frac{dv}{dx}\right)\delta v\right\}$$

$$-\iint dx\, dz \left\{ B\delta v + B\left(\frac{du}{dy}\delta u + \frac{dv}{dy}\delta v + \frac{dw}{dy}\delta w\right)\right.$$

$$\left. + \left(R\frac{du}{dx} + H\frac{dv}{dy} + P\frac{dw}{dz}\right)\delta v + L\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\delta w + N\left(\frac{du}{dy} + \frac{dv}{dx}\right)\delta u\right\}$$

$$-\iint dx\, dy \left\{ C\delta w + C\left(\frac{du}{dz}\delta u + \frac{dv}{dz}\delta v + \frac{dw}{dz}\delta w\right)\right.$$

$$\left. + \left(Q\frac{du}{dx} + P\frac{dv}{dy} + I\frac{dw}{dz}\right)\delta w + L\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\delta v + M\left(\frac{du}{dz} + \frac{dw}{dx}\right)\delta u\right\}$$

$$+ \iiint dx\, dy\, dz\, \delta u \left\{(G + A)\frac{d^2u}{dx^2} + (N + B)\frac{d^2u}{dy^2} + (M + C)\frac{d^2u}{dz^2}\right.$$

$$\left. + (R + N)\frac{d^2v}{dx\, dy} + (Q + M)\frac{d^2w}{dx\, dz}\right\}$$

$$+ \iiint dx\, dy\, dz\, \delta v \left\{(N + A)\frac{d^2v}{dx^2} + (H + B)\frac{d^2v}{dy^2} + (L + C)\frac{d^2v}{dz^2}\right.$$

$$\left. + (N + R)\frac{d^2u}{dx\, dy} + (P + L)\frac{d^2w}{dy\, dz}\right\}$$

$$+ \iiint dx\, dy\, dz\, \delta w \left\{(M + A)\frac{d^2w}{dx^2} + (L + B)\frac{d^2w}{dy^2} + (I + C)\frac{d^2w}{dz^2}\right.$$

$$\left. + (M + Q)\frac{d^2u}{dx\, dz} + (L + P)\frac{d^2v}{dy\, dz}\right\}.$$

Neglecting the double integrals which relate to the extreme boundaries only of the medium, and which we will suppose situated at an infinite distance, we get for the general equations of motion,

$$\rho \frac{d^2u}{dt^2} = (G + A)\frac{d^2u}{dx^2} + (N + B)\frac{d^2u}{dy^2} + (M + C)\frac{d^2u}{dz^2} + (R + N)\frac{d^2v}{dx\,dy} + (Q + M)\frac{d^2w}{dx\,dz},$$

$$\rho \frac{d^2v}{dt^2} = (N + A)\frac{d^2v}{dx^2} + (H + B)\frac{d^2v}{dy^2} + (L + C)\frac{d^2v}{dz^2} + (N + R)\frac{d^2u}{dx\,dy} + (P + L)\frac{d^2w}{dy\,dz}, \qquad (5.)$$

$$\rho \frac{d^2w}{dt^2} = (M + A)\frac{d^2w}{dx^2} + (L + B)\frac{d^2w}{dy^2} + (I + C)\frac{d^2w}{dz^2} + (M + Q)\frac{d^2u}{dx\,dz} + (L + P)\frac{d^2v}{dy\,dz}.$$

If now in our indefinitely extended medium we wish to determine the laws of the propagation of plane waves, we must take, to satisfy the last equations,

$$u = \alpha f(ax + by + cz + et),$$

$$v = \beta f(ax + by + cz + et),$$

$$w = \gamma f(ax + by + cz + et);$$

a, b and c being the cosines of the angles which a normal to the wave's front makes with the co-ordinate axes, α, β, γ constant coefficients, and e the velocity of transmission of a wave perpendicular to its own front, and taken with a contrary sign.

Substituting these values in the equations (5), and making to abridge

$$A' = (G + A)a^2 + (N + B)b^2 + (M + C)c^2,$$

$$B' = (N + A)a^2 + (H + B)b^2 + (L + C)c^2,$$

$$C' = (M + A)a^2 + (L + B)b^2 + (I + C)c^2;$$

$$D' = (L + P)\,bc,$$
$$E' = (M + Q)\,ac,$$
$$F' = (N + R)\,ab;$$

we get

$$\begin{aligned} 0 &= (A' - e^2)\,\alpha + F'\beta + E'\gamma, \\ 0 &= F'\alpha + (B' - e^2)\,\beta + D'\gamma, \\ 0 &= E'\alpha + D'\beta + (C - e^2)\,\gamma; \end{aligned} \qquad (6.)$$

These last equations will serve to determine three values of e^2, and three corresponding ratios of the quantities α, β, γ; and hence we know the directions of the disturbance by which a plane wave will propagate itself without subdivision, and also the corresponding velocities of propagation. From the form of the equations (6), it is well known, that if we conceive an ellipsoid whose equation is

$$1 = A'x^2 + B'y^2 + C'z^2 + 2D'yz + 2E'xz + 2F'xy,^* \qquad (7.)$$

and represent its three semi-axes by r', r'', and r''', the directions of these axes will be the required directions of the disturbance, and the corresponding velocities of propagation will be given by

$$e^2 = \frac{1}{r^2}.$$

Fresnel supposes those vibrations of the particles of the luminiferous ether which affect the eye, to be *accurately* in the front of the wave.

* If we reflect on the connexion of the operations by which we pass from the function (4) to the equation (7), it will be easy to perceive that the right side of the equation (7) may always be immediately deduced from that portion of the function which is of the second degree by changing u, v and w into x, y and z.

Also, $\frac{d}{dx}$, $\frac{d}{dy}$ and $\frac{d}{dz}$ into a, b and c.

This remark will be of use to us afterwards, when we come to consider the most general form of the function due to the internal actions.

Let us therefore investigate the relation which must exist between our coefficients, in order to satisfy this condition for two of our three waves, the remaining one in consequence being necessarily propagated by normal vibrations.

For this we may remark, that the equation of a plane parallel to the wave's front is

$$0 = ax' + by' + cz'. \qquad (a)$$

If therefore we make

$$x = x' + a\lambda,$$
$$y = y' + b\lambda,$$
$$z = z' + c\lambda,$$

and substitute these values in the equation (7) of the ellipsoid: restoring the values of

$$A', B', C', D', E', F',$$

the odd powers of λ ought to disappear in consequence of the equation (a), whatever may be the position of the wave's front. We thus get

$$G = H = I = \mu \qquad \text{suppose,}$$

and

$$P = \mu - 2L,$$
$$Q = \mu - 2M,$$
$$R = \mu - 2N.$$

In fact, if we substitute these values in the function (4) there will result

$$-2\phi = -2\phi_1 - 2\phi_2 =$$

$$2A\frac{du}{dx} + 2B\frac{dv}{dy} + 2C\frac{dw}{dz}$$

$$+ A\left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right\}$$

$$+ B\left\{\left(\frac{du}{dy}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dy}\right)^2\right\}$$

$$+ C\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2 + \left(\frac{dw}{dz}\right)^2\right\} \qquad \text{(A)}$$

$$+\mu\left\{\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)^2\right.$$

$$+L\left\{\left(\frac{dv}{dz}+\frac{dw}{dy}\right)^2-4\frac{dv}{dy}\frac{dw}{dz}\right\}$$

$$+M\left\{\left(\frac{du}{dz}+\frac{dw}{dx}\right)^2-4\frac{du}{dx}\frac{dw}{dz}\right\}$$

$$+N\left\{\left(\frac{du}{dy}+\frac{dv}{dx}\right)^2-4\frac{du}{dx}\frac{dv}{dy}\right\},$$

which, when $0=A$, $0=B$, $0=C$, reduces to the last four lines.

Making the same substitution in the equation (7), we get

$$\begin{aligned}1 &= \mu(ax+by+cz)^2,\\ &+(Aa^2+Bb^2+Cc^2)(x^2+y^2+z^2),\\ &+L(cy-bz)^2+M(az-cx)^2+N(bx-ay)^2.\end{aligned} \qquad (8.)$$

Let us in the first place suppose the system free from all extraneous pressure.

Then $A=0$, $B=0$, $C=0$,

and the above equation combined with that of a plane parallel to the wave's front will give

$$0=ax+by+cz \qquad (9.)$$

$$1=L(cy-bz)^2+M(az-cx)^2+N(bx-ay)^2,$$

the equations of an infinite number of ellipses which in general do not belong to the same curve surface. If, however, we cause each ellipsis to turn 90^0 in its own plane, the whole system will belong to an ellipsoid, as may be thus shewn: Let (xyz) be the co-ordinates of any point p in its original position, and $(x'y'z')$ the co-ordinates of the point p' which would coincide with p when the ellipse is turned 90^0 in its own plane. Then

$$x^2+y^2+z^2=x'^2+y'^2+z'^2,$$

since the distance from the origin O is unaltered;

$$0=ax'+by'+cz', \text{ since the plane is the same;}$$

$$0=xx'+yy'+zz', \text{ since } pOp'=90^0.$$

The two last equations give

$$\frac{x'}{cy - bz} = \frac{y'}{az - cx} = \frac{z'}{bx - ay} = \omega \text{ suppose.}$$

Hence the last of the equations (9) becomes

$$\omega^2 = Lx'^2 + My'^2 + Nz'^2.$$

But

$$x'^2 + y'^2 + z'^2 = \omega^2 \{(cy - bz)^2 + (az - cx)^2 + (bx - ay)^2\},$$

$$= \omega^2 \{(b^2 + a^2) z^2 + (c^2 + a^2) y^2 + (b^2 + c^2) x^2 - 2 (bcyz + abxy + acxz)\},$$

$$= \omega^2 \{(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2\},$$

$$= \omega^2 (x^2 + y^2 + z^2) = x^2 + y^2 + z^2,$$

$$\therefore \ \omega^2 = 1,$$

and our equation finally becomes

$$1 = Lx'^2 + My'^2 + Nz'^2. \qquad (10.)$$

We thus see, that if we conceive a section made in the ellipsoid to which the equation (10) belongs, by a plane passing through its centre and parallel to the wave's front, this section, when turned 90 degrees in its own plane, will coincide with a similar section of the ellipsoid to which the equation (8) belongs, and which gives the directions of the disturbance that will cause a plane wave to propagate itself without subdivision, and the velocity of propagation parallel to its own front. The change of position here made in the elliptical section, is evidently equivalent to supposing the actual disturbances of the ethereal particles to be parallel to the plane usually denominated the *plane of polarization*.

This hypothesis, at first advanced by M. Cauchy, has since been adopted by several philosophers; and it seems worthy of remark, that if we suppose an elastic medium free from all extraneous pressure, we have merely to suppose it so constituted that two of the wave-disturbances shall be *accurately* in the wave's front, agreeably to Fresnel's fundamental hypothesis, thence to deduce his general construction for the propagation of waves in biaxal crystals. In fact, we shall afterwards prove that the function ϕ_2, which in its most general form contains twenty-one coefficients, is, in consequence of this hypothesis, reduced to one containing only seven coefficients; and that, from this last form of our function, we obtain for the directions of the disturbance and velocities of propagation precisely the same values as given by Fresnel's construction.

The above supposes, that in a state of equilibrium every part of the medium is quite free from pressure. When this is not the case, A B and C will no longer vanish in the equation (8). In the first place, conceive the plane of the wave's front parallel to the plane (yz); then $a = 1$, $b = 0$, $c = 0$, and the equation (8) of our ellipsoid becomes

$$1 = \mu x^2 + A(x^2 + y^2 + z^2) + Mz^2 + Ny^2;$$

and that of a section by a plane through its centre parallel to the wave's front, will be

$$1 = (A + N)y^2 + (A + M)z^2;$$

and hence, by what precedes, the velocities of propagation of our two polarized waves will be

$\sqrt{A + N}$. The disturbance being parallel to the axis of y.

$\sqrt{A + M}$. .... to the axis of z.

Similarly, if the plane of the wave's front is parallel to the plane (xz), the wave-velocities are,

$\sqrt{B + N}$. The disturbance being parallel to the axis x.

$\sqrt{B + L}$. .... to the axis z.

Or, if the plane of the wave's front is parallel to (xy), the velocities are,

$\sqrt{C + M}$. The disturbance being parallel to x.

$\sqrt{C + L}$. .... y.

Fresnel supposes that the wave-velocity depends on the direction of the disturbance only, and is independent of the position of the wave's front. Instead of assuming this to be generally true, let us merely suppose it holds good for these three principal waves. Then we shall have

$$N + A = C + L, \quad M + A = B + L \text{ and } B + N = C + M;$$

or, we may write

$$A - L = B - M = C - N = \nu. \quad \text{(Suppose.)}$$

Thus our equation (8) becomes, since $a^2 + b^2 + c^2 = 1$,

$$\begin{aligned} 1 &= \mu\,(ax + by + cz)^2 + \nu\,(x^2 + y^2 + z^2) \\ &+ (La^2 + Mb^2 + Nc^2)\,(x^2 + y^2 + z^2) \\ &+ L\,(cy - bz)^2 + M\,(az - cx)^2 + N\,(bx - ay)^2. \end{aligned}$$

But the two last lines of this formula easily reduce to

$$\begin{aligned} &(M + N)\,x^2 + (N + L)\,y^2 + (L + M)\,z^2 \\ &+ L\,\{a^2 x^2 - (by + cz)^2\} + M\,\{b^2 y^2 - (ax + cz)^2\} \\ &+ N\,\{c^2 z^2 - (ax + by)^2\}, \end{aligned}$$

and hence our last equation becomes

$$\left.\begin{aligned} 1 = (\nu + M + N)\,x^2 + (\nu + N + L)\,y^2 + (\nu + L + M)\,z^2 \\ + \mu\,(ax + by + cz)^2 \\ + L\,\{a^2x^2 - (by + cz)^2\} \\ + M\,\{b^2y^2 - (ax + cz)^2\} \\ + N\,\{c^2z^2 - (ax + by)^2\}. \end{aligned}\right\} \quad (11.)$$

In consequence of the condition which was satisfied in forming the equation (8), it is evident that two of its semi-axes are in a plane parallel to the wave's front, and of which the equation is

$$0 = ax + by + cz, \qquad (12.)$$

the same therefore will be true for the ellipsoid whose equation is (11), as this is only a particular case of the former. But the section of the last ellipsoid by the plane (12) is evidently given by

$$1 = (\nu + M + N)\,x^2 + (\nu + L + N)\,y^2 + (\nu + L + M)\,z^2,$$
$$0 = ax + by + cz. \qquad (12,\ 1.)$$

By what precedes, the two axes of this elliptical section will give the two directions of disturbance which will cause a wave to be propagated without subdivision, and the velocity of propagation of each wave will be inversely as the corresponding semi-axis of the section; which agrees with Fresnel's construction, supposing, as he has done, the actual direction of the disturbance of the particles of the ether is perpendicular to the plane of polarization.

Let us again consider the system as quite free from extraneous pressure, and take the most general value of ϕ_2 containing twenty-one coefficients. Then, if to abridge, we make

$$\frac{du}{dx} = \xi, \quad \frac{dv}{dy} = \eta, \quad \frac{dw}{dz} = \zeta;$$

$$\frac{dv}{dz} + \frac{dw}{dy} = a, \quad \frac{du}{dz} + \frac{dw}{dx} = \beta, \quad \frac{du}{dy} + \frac{dv}{dx} = \gamma,$$

we shall have

$$\begin{aligned} -\phi_2 = {} & (\xi^2)\,\xi^2 + (\eta^2)\,\eta^2 + (\zeta^2)\,\zeta^2 + 2(\eta\zeta)\,\eta\zeta + 2(\xi\zeta)\,\xi\zeta + 2(\xi\eta)\,\xi\eta \\ & + (a^2)\,a^2 + (\beta^2)\,\beta^2 + (\gamma^2)\,\gamma^2 + 2(\beta\gamma)\,\beta\gamma + 2(a\gamma)\,a\gamma + 2(a\beta)\,a\beta \\ & + 2(a\xi)\,a\xi + 2(\beta\xi)\,\beta\xi + 2(\gamma\xi)\,\gamma\xi \\ & + 2(a\eta)\,a\eta + 2(\beta\eta)\,\beta\eta + 2(\gamma\eta)\,\gamma\eta \\ & + 2(a\zeta)\,a\zeta + 2(\beta\zeta)\,\beta\zeta + 2(\gamma\zeta)\,\gamma\zeta, \end{aligned}$$

where $(\xi^2)\,(a^2)$, &c., are the twenty-one coefficients which enter into ϕ_2.

Suppose now the equation to the front of a wave is

$$0 = ax + by + cz.$$

Then, by what was before observed, the right side of the equation of the ellipsoid, which gives the directions of disturbance of the three polarized waves and their respective velocities, will be had from ϕ_2, by changing u, v and w into x, y and z;

$$\text{also } \frac{d}{dx}, \ \frac{d}{dy} \text{ and } \frac{d}{dz} \text{ into } a, \ b \text{ and } c.$$

We shall thus get

$$1 = Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy.$$

Provided

$$A = (\xi^2)\,a^2 + (\beta^2)\,c^2 + (\gamma^2)\,b^2 + 2(\beta\gamma)\,bc + 2(\xi\beta)\,ac + 2(\xi\gamma)\,ab,$$

$$B = (\eta^2)\,b^2 + (a^2)\,c^2 + (\gamma^2)\,a^2 + 2(a\gamma)\,ac + 2(\eta a)\,bc + 2(\eta\gamma)\,ab,$$

$$C = (\zeta^2)\,c^2 + (a^2)\,b^2 + (\beta^2)\,a^2 + 2(a\beta)\,ab + 2(\zeta a)\,bc + 2(\zeta\beta)\,ac,$$

$$D = (\eta\zeta)\,bc + (a^2)\,bc + (\beta\gamma)\,a^2 + (a\beta)\,ac + (a\gamma)\,ab$$
$$+ (a\eta)\,b^2 + (a\zeta)\,c^2 + (\beta\eta)\,ab + (\gamma\zeta)\,ac,$$

$$E = (\xi\zeta)\,ac + (\beta^2)\,ac + (a\gamma)\,b^2 + (a\beta)\,bc + (\beta\gamma)\,ab$$
$$+ (\beta\xi)\,a^2 + (\beta\zeta)\,c^2 + (a\xi)\,ab + (\gamma\zeta)\,bc,$$

$$F = (\xi\eta)\,ab + (\gamma^2)\,ab + (a\beta)\,c^2 + (a\gamma)\,bc + (\beta\gamma)\,ac$$
$$+ (\gamma\xi)\,a^2 + (\gamma\eta)\,b^2 + (a\xi)\,ac + (\beta\eta)\,bc$$

But if the directions of two of the disturbances are rigorously in the front of a wave, a plane parallel to this front passing through the center of the ellipsoid, and whose equation is

$$0 = ax + by + cz,$$

must contain two of the semi-axes of the ellipsoid; and therefore a system of chords perpendicular to this plane will be bisected by it; and hence we get

$$0 = (A - C)\,ac + E\,(c^2 - a^2) + Fbc - Dab,$$

$$0 = (B - C)\,bc + D\,(c^2 - b^2) + Fac - Eab.$$

Substituting in these the values of A, B, &c., before given, we shall obtain the fourteen relations following between the coefficients of ϕ_2, viz.

$$0 = (\alpha\eta), \quad 0 = (\beta\xi), \quad 0 = (\gamma\xi),$$

$$0 = (\alpha\zeta), \quad 0 = (\beta\zeta), \quad 0 = (\gamma\eta),$$

$$(\alpha\xi) = -2(\beta\gamma), \quad (\beta\eta) = -2(\alpha\gamma), \quad (\gamma\zeta) = -2(\alpha\beta),$$

$$(\xi^2) = (\eta^2) = (\zeta^2) = 2(\alpha^2) + (\eta\zeta) = 2(\beta^2) + (\xi\zeta) = 2(\gamma^2) + (\xi\eta).$$

Hence, we may readily put the function ϕ_2 under the following form:

$$(\xi^2)(\xi + \eta + \zeta)^2 + (\alpha^2)\{\alpha^2 - 4\eta\zeta\} + (\beta^2)(\beta^2 - 4\xi\zeta)$$
$$+ (\gamma^2)(\gamma^2 - 4\xi\eta) + 2(\beta\gamma)(\beta\gamma - 2\alpha\xi) \ldots\ldots\ldots\ldots$$
$$+ 2(\alpha\gamma)(\alpha\gamma - 2\beta\eta) + 2(\alpha\beta)(\alpha\beta - 2\gamma\zeta),$$

or by restoring the values of ξ, η, &c., and making $G = (\xi^2)$ $L = (\alpha^2)$ &c., our function will become

$$G\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)^2 + L\left\{\left(\frac{dv}{dz} + \frac{dw}{dy}\right)^2 - 4\frac{dv}{dy}\frac{dw}{dz}\right\}$$
$$+ M\left\{\left(\frac{du}{dz} + \frac{dw}{dx}\right)^2 - 4\frac{du}{dx}\frac{dw}{dz}\right\} + N\left\{\left(\frac{du}{dy} + \frac{dv}{dx}\right)^2 - 4\frac{du}{dx}\frac{dv}{dy}\right\}$$
$$+ 2P\left\{\left(\frac{du}{dz} + \frac{dw}{dx}\right)\left(\frac{du}{dy} + \frac{dv}{dx}\right) - 2\frac{du}{dx}\left(\frac{dv}{zd} + \frac{dw}{dy}\right)\right\}$$
$$+ 2Q\left\{\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\left(\frac{du}{dy} + \frac{dv}{dx}\right) - 2\frac{dv}{dy}\left(\frac{du}{dz} + \frac{dw}{dx}\right)\right\} \qquad (12.)$$
$$+ 2R\left\{\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\left(\frac{du}{dz} + \frac{dw}{dx}\right) - 2\frac{dw}{dz}\left(\frac{du}{dy} + \frac{dv}{dx}\right)\right\},$$

and hence we get for the equation of the corresponding ellipsoid,

$$1 = G(ax + by + cz)^2 + L(bz - cy)^2$$
$$+ M(az - cx)^2 + N(ay - bx)^2 + 2P(cx - az)(ay - bx) \qquad (13.)$$
$$+ 2Q(bz - cy)(ay - bx) + 2R(bz - cy)(cx - az).$$

But if in equation (8) and corresponding function (A), we suppose $A = 0$, $B = 0$ and $C = 0$, and then refer the equation to axes taken arbitrarily in space, we shall thus introduce three new coefficients, and evidently obtain a result equivalent to equation (13) and function (12). We therefore see that the single supposition of the wave-disturbance, being always *accurately* in the wave's front, leads to a result equivalent to that given by the former process; and we are thus assured that by employing the simpler method we do not, in the case in question, eventually lessen the generality of our result, but merely, in effect, select the three rectangular axes, which may be called the axes of elasticity of the medium for our co-ordinate axes. From the general form of ϕ_1 it is clear that the same observation applies to it, and therefore the consequences before deduced possess all the requisite generality.

The same conclusions may be obtained, whether we introduce the consideration of extraneous pressures or not, by direct calculation. In fact, when these pressures vanish, and we conceive a section of the ellipsoid whose equation is (13) made by a plane parallel to the wave's front, to turn 90 degrees in its own plane, the same reasoning by which equation (10) was before found, immediately gives, in the present case,

$$1 = Lx'^2 + My'^2 + Nz'^2 + 2Py'z' + 2Qx'z' + 2Rx'y' \qquad (14.)$$

for the equation of the surface in which all the elliptical sections in their new situations, and corresponding to every position of the wave's front, will be found.

Lastly, when we introduce the consideration of extraneous pressures, it is clear, from what precedes, that we shall merely have to add to the function on the right side of the equation (13) the quantity

$$(Aa^2 + Bb^2 + Cc^2 + 2Dbc + 2Eac + 2Fab)(x^2 + y^2 + z^2),$$

which would arise from changing u, v and w into x, y and z. Also $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ into a, b, c, in that part of ϕ_1 which is of the second degree in u, v, w, agreeably to the remark in a foregoing note. Afterwards, when we determine the values of A, B, &c., by the same condition which enabled us to deduce the system (12, 1), we shall have, in the place of this system, the following:

$$1 = K(x^2 + y^2 + z^2) - \{Lx^2 + My^2 + Nz^2 + 2Pyz + 2Qxz + 2Rxy\} \ldots \quad (15.)$$

$$0 = ax + by + cz,$$

which is applicable to the more general case just considered.

VIII. *On a Portion of the Tertiary Formations of Switzerland.* By D. T. Ansted, Esq. M.A., *Fellow of Jesus College. Fellow of the Society and of the Geological Society; Professor of Geology in King's College, London.*

[Read *May* 20, 1839.]

The Tertiary formations of Switzerland are singularly deficient in most of those points which have rendered the contemporaneous deposits in other countries of Europe so attractive and important. The beds, for the most part, vary but little in mineral structure: they seem to have been accumulated rapidly, and under circumstances little favourable to the preservation of organic remains, and the few fossils that are known to occur, possess none of that definite character, which elsewhere indicates with sufficient clearness to what well-known group the one in question was anterior, and what beds were anterior to it. Owing, perhaps, to this want of determinate character, and partly, also, no doubt, to the superior interest of the strangely contorted secondary beds, which form the principal mass of the great mountain district always within sight, it has happened that travellers in general have neglected to examine carefully the great valley of Switzerland, and I am not aware of any detailed account in our own language of so considerable a portion of European Tertiary Geology.

I am not able, indeed, myself to add much to the small amount of our knowledge on this subject, but anxious at all events to direct attention to it, I have ventured to lay before the Society a few observations made during a stay of several weeks at Lausanne, in the summer of 1838. In order to do this most effectually, I shall first consider the nature of the Tertiary beds occurring in what is called the Great Helvetic Basin, and occupying the space between the High Alps and the Jura chain. I shall afterwards proceed to remark upon the various smaller basins met with in the Jura district itself, and partially filling up the valleys between the different ranges of that mountain chain.

From whichever side Switzerland is entered, whether from France, Germany, or Italy, no traveller, not even the most indifferent about geological phenomena, can have failed to notice the physical structure of the country, or the effect to the eye of that series of deposits concerning which I am about to speak. The high range of mountains to the South, nearly terminated at each end by the two highest of the European mountains, Mont Blanc, and Monte Rosa—the continuation of these lofty eminences toward the North-East, forming the "High Alps," and extending into the Northern Cantons of Switzerland—the less lofty but still considerable elevations running parallel to this principal range in the West of Switzerland towards France, and known as the "Jura" chain—all these very remarkable and strikingly beautiful mountain chains surround a tract of land comparatively level and rich in every thing that can administer to the wants or luxuries of man; and it is this cultivated district, this comparative plain in a land of mountains, which marks out the extent of the Swiss Tertiary deposits, and has hitherto been, as I observed, almost neglected by the geologist. It requires, perhaps, to have been on the spot to understand the temptation offered by the near proximity of *such* mountains; but those who have been there, and have hurried on with all the enthusiasm and excitement of novelty to breathe the pure and exhilirating mountain air, will wonder but little that the plains have been neglected, and that the Tertiary Geology has given place to the Alpine.

It was hardly an effort of philosophy which induced me to labour in the less trodden field:—a conviction that I could not hope to make much way where so many and far superior and more practised geologists had preceded me, may indeed have induced me the more readily to be contented in a less distinguished sphere, but my expeditions from Lausanne were necessarily short, and my opportunities limited.

Thus circumstanced, my observations will be found to relate chiefly to the South-Western part of the Helvetic Basin, and not at all to the more interesting portion extending Northwards and Eastwards from Berne, and already somewhat minutely described in a work, published in 1825, by Professor Studer, of Berne. My excursions were, as I have said, confined to a small part of the Canton of Freyburg, and the greater part of the Pays de Vaud. The limit of this district to the South is the lake of Geneva. The Eastern and Western boundaries are sufficiently defined by the abrupt elevation of mountains, forming the flanks of the High Alps on the one side, and of the Jura on the other.

The high road from Freyburg to Vevey is nowhere at any great distance from the line which separates the tertiary beds from those secondary ones upon which they lie uncomformably, but the actual junction at any point I did not perceive, as the country is for the most part covered up, and the geological phenomena obliterated. Close to Vevey, however, in a valley cut by a small stream coming down to the lake, we obtain a glimpse of the extreme tertiary beds to the East, and it will be perhaps best if, commencing with these, we trace the collocation of the beds as they are exposed on the North side of the lake of Geneva, and may be observed in travelling from Vevey towards Lausanne and Geneva, westwards.

Close to the town of Vevey there occurs a hard conglomerate, very coarse where it rests on the older rock below, but becoming gradually finer, until after a few miles it is replaced by a very fine sandstone, which spreads over the whole centre of the valley of Switzerland, and is the great tertiary deposit of which I have chiefly to speak. Of these beds, the coarser conglomerate is known generally by its German designation, "Nagelfluhe," while the nature and peculiarities of the finer sandstone (which is the most widely spread and extensive of all the European Tertiaries) are indicated in the name "Molasse," by which the soft, incoherent tertiary sandstone of this country and Germany is designated.

The thickness of the Nagelfluhe is various, but never very great. From near Vevey it may be traced towards the West for about a couple of miles, gradually becoming a finer deposit, and imperceptibly changing into the Molasse, without any definite line of separation.

It would be extremely difficult to lay down the limits of this bed with accuracy, although the thickness cannot be any where very great; I could not discover a single spot where the dip could be taken, but as the whole seems to have undergone a change of position by disturbances connected, doubtless, with the uplifting of the mountain chain, no single observation of this kind, even if it were made, could possess much value in the way of determining the mass of the deposit.

On the other hand, the thickness of the Molasse, although equally difficult to determine, must be enormous, and if calculated in the ordinary way, allowing for its being repeated once or twice by faults, will still appear almost incredible. Extending across the valley of Switzerland for nearly five and twenty miles, and inclined often at angles varying from 15 to more than 50 degrees, rising sometimes into hills four thousand feet above the sea-level and more than two thousand above the general level of the country, we cannot escape the conclusion that it is a mass of vast thickness, even after making every allowance for the effects of disturbance.

I am inclined, however, to think that much of this appearance of enormous thickness is owing to the deposit having been formed on a considerable slope, and not on a horizontal or nearly horizontal plain, and that thus its almost uniform inclination is not owing entirely to disturbances of the substratum, but also to the circumstances of deposition. If we imagine the formation to have been commenced when the level of the valley of Switzerland was below that of the sea, and that sandbanks rapidly formed on a shelving coast at some distance from the shore, were gradually raised by successive small elevations, and afterwards when the general level of the land was above that of the sea, that the elevations had gone on from time to time till the present state of things was produced, we should have very similar phenomena presented to view, viz. an enormous mass of sandstone, appearing to possess a dip that would multiply its real thickness tenfold, and ranges of hills at some distance from the former coast-line.

The main difference between the Nagelfluhe and the Molasse, consists in the mechanical difference between a coarse conglomerate and a fine sandstone, but interstratified with the Molasse there occur here and there beds of lignite, which add much to the geological interest and something to the economical advantages of the district under consideration. There is also found in the West of the Canton of Vaud, not far from the lake of Neuchâtel, a white building-stone containing much calcareous matter in its composition, but circumstances prevented me from paying that attention to so interesting a stratum which it well deserves from the geologist. It will be found forming a hill close to the little town of Thierrens, and I observed it in one spot dipping about 40 degrees to the South-West*.

Having thus described the mineral composition of the different strata observed, I come now to speak of the general outline of the country, and the deductions to be drawn from considering the physical features produced probably by disturbances acting after the beds had been deposited.

Although, in comparison with the stupendous chain of the Alps, the central and more cultivated portion of Switzerland is properly designated as a valley, yet even in this valley there occur eminences which in a more level country might well be called mountains. About five miles from Vevey, and to the west of the coarse conglomerate called Nagelfluhe, there rises a hill of Molasse to the height of nearly 4000 feet, and a chain of hills may be observed extending from this (which is called the Tour de Gourze) towards the North-East, whose heights are successively, 3000, 4000, and 3500 feet above the sea. In speaking of these altitudes, however, it must not be forgotten, that the level of the lakes of Geneva and Neuchâtel is considerably more than twelve hundred feet above the sea, and thus the hills do not in reality form such striking features in the landscape as others of no greater actual elevation, but rising from a lower plateau, in other countries, and under different circumstances. Imbedded in the sandstone of which these hills are composed, there occurs in the line of the hills, and about ten miles North of Vevey, one of the beds of lignite already alluded to, and we are enabled accordingly to determine the dip with some accuracy, at all events in this spot; I observed that it was very considerable, certainly more than 50°, and its direction variable, though on the whole Easterly, being here, and in one or two other places along the line, towards the South-East, in a few others North-East, and sometimes nearly due East.

* This would appear to be a local deviation from the general dip of the district.

If leaving the chain of hills just alluded to we advance along the banks of the lake of Geneva, towards the West, we come to a parallel but less elevated chain, beginning about ten miles from Vevey, near the town of Lausanne, forming a ridge of sand-hills whose summits are about 2500 feet above the level of the sea, and the ridge continues at nearly the same elevation for a distance of at least 15 or 20 miles to N. E. Here, as before, the dip is towards the South-East, and generally as much as 45°.

Between Vevey and Lausanne another bed of lignite of some thickness is worked. The bed is exposed in consequence of a mountain torrent having cut its way through the Molasse, close to the spot where the lignite crops out to the surface. It is thus worked in chambers, from the right bank of the stream to the outcrop, which is at no great distance.

If we return now, and continue our course along the banks of the lake still further to the West, we shall find a third time indications of a similar North and South range, commencing at a celebrated *point de vue*, called the Signal of Bougi, from which may be enjoyed one of the most beautiful and picturesque prospects in this part of Switzerland. The chain of hills commencing here, is continued at an elevation of little less than 3000 feet for many miles, parallel to the mountains of the Jura.

In several places the dip of the Molasse may be observed in the neighbourhood of this, as of the other parallel lines of elevation, and is generally South-East. The opportunities however of obtaining dips are so very rare, and except where the lignite occurs, the bedding so obscure, that if it were not for the uniformity wherever the inclination can be clearly made out, I could hardly venture to lay much stress on a series of observations, so few in comparison with the large extent of country over which they are spread.

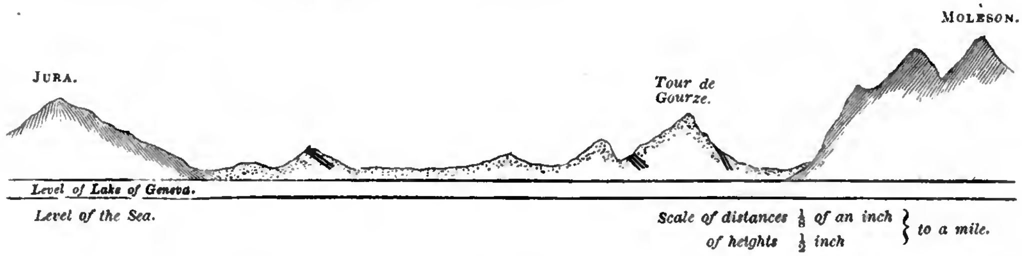
On the whole, however, we seem to have in this Southern portion of the Molasse of Switzerland, three distinct and tolerably well-marked lines of elevation, all parallel to the mountain chain of the Jura, *from* which also they all dip. The upheaving of this latter chain (the Jura) subsequent to the formation of the High Alps, seems to have been the means by which the peculiar physical features of these tertiary beds were in a great measure produced. Doubtless there have been great changes effected by the action of the elements upon beds so soft, and often almost incoherent; but still the great amount of dip considered in connection with the parallel ranges alluded to, gives us sufficient reason for referring to elevations as the original causes of the more remarkable phenomena.

The Tertiary Geology of Switzerland is but little assisted by the consideration of those organic remains which are peculiar to, or discovered in the various beds. The Molasse is so exceedingly barren of fossils, that during many weeks which I spent in the immediate neighbourhood of great natural sections of it, I did not on any occasion find a single specimen indicating organic structure. The Northern beds are, however, rather more prolific, and offer sufficient evidence that this vast mass of sand was accumulated under sea-water. There is a list of fossils in the work by Professor Studer, already alluded to, which includes the following marine genera,—Mactra, Cytherea, Cardium, Pecten, Trochus, Cassis, Terebra, Buccinum, and Conus. These were most of them found in various parts of the Cantons of Zurich and Lucerne.

Although, however, the general character of the bed, as well as the discovery of such a series of fossils, would induce us to place the whole formation among marine deposits, yet with regard to the bands of lignite, the evidence is so entirely the other way, and points so clearly to a fresh-water origin, that I think the only way of reconciling the apparent anomaly is to suppose the former existence of considerable streams rushing down from the mountains, and bringing with them vast quantities of vegetable, intermixed with some animal remains, which might be deposited at the mouth of a river in consequence of a bar, or extensive sandbank.

The shells found in the lignite, and embedded in the sandstone immediately adjacent, are chiefly Helix, Planorbis, Lymnœa, and Unio; but the specimens are so much broken, that the exact species can hardly be determined. Besides these, I was fortunate enough, on one occasion, to discover a portion of the sternum of a chelonian reptile, probably a turtle, although such fossils are, I believe, extremely rare, and I did not hear of any other remains of Reptiles during my stay in the South of Switzerland. The lignite is generally hard with a clean conchoidal fracture and brilliant lustre, and is a good deal used for fuel. It is met with in beds several feet in thickness, but not extending far in any direction; and these beds alternate usually with thin marls, which are often quite white in consequence of the enormous abundance of crushed shells belonging to land and fresh-water species, which often completely hide the marl, and cover the surface of the lignite.

SECTION I. ACROSS THE GREAT VALLEY OF SWITZERLAND.



From the annexed section some idea may be formed of the relative positions and magnitude of the three lines of elevation already alluded to as existing in the Molasse, and the place which the two beds of lignite occupy: the dips are principally towards the East, and more or less with a Southerly tendency, but the amount varies, and is, I think, generally most considerable in the neighbourhood of the High Alps, *towards* which the strata incline.

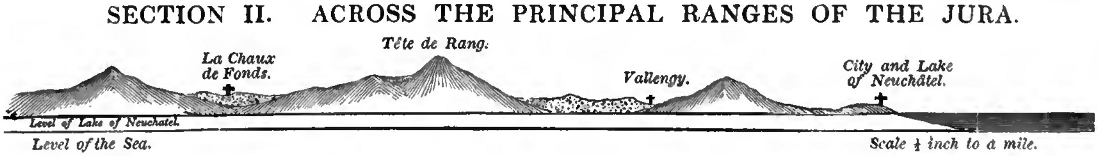
I have now only to add a few words more on this part of my subject; viz. to point out, so far as I am able, what remains to be done for the more complete illustration of the Tertiary Geology of Switzerland.

In the first place, there occurs a question of great interest, and one which requires, probably, very accurate research to determine: viz. whether the lines of elevation to which I have directed attention were really caused by upheaving forces, or merely by denudation—whether, in a word, there are lines of fault, or anticlinal axes, corresponding to the lines of elevation. In the next place, it would be extremely interesting to identify, if possible, the two beds of lignite—a task which I was unable to perform; and lastly, it is possible that, barren as the sandstone is of fossils, some may yet be discovered, by which we may declare with certainty the actual geological age of the formation. With regard to this latter point, I shall have a few words to add at the conclusion of this paper, and must for any further information refer to the work already alluded to, published at Berne by M. Studer.

Quitting the wide expanse of the great Helvetic Basin, I wish next to direct attention to the circumstances connected with the tertiary valleys of the Jura, and more particularly to the valley of la Chaux de Fonds, which may serve indeed as a type of the rest.

The villages of la Chaux de Fonds and le Locle, at the two extremities of the same valley, are the richest, the most populous, and, in some respects, the most remarkable of any in Switzerland. They are situated near the frontier of France, one in the Northern and the other in the Southern part of a valley which is about ten miles long and one broad, extending in a North-Easterly direction, at an elevation of more than two thousand feet above the level of the sea. There is no outlet to the valley for drainage at either extremity, and its general appearance, as well as geological structure, show clearly that it was formerly the bed of a mountain lake, resembling in all probability those still existing in the Jura, such as the Lac de Joux, the Lac de St. Point, and one or two others. As I first visited la Chaux de Fonds from Neuchâtel, and afterwards entering the valley at its South-Western extremity, passed le Locle and again reached the village on my journey Northwards, I will first describe in a few words the section across the Jura, and then the peculiarities which present themselves in tracing the beds in the direction of the valley's length.

SECTION II. ACROSS THE PRINCIPAL RANGES OF THE JURA.

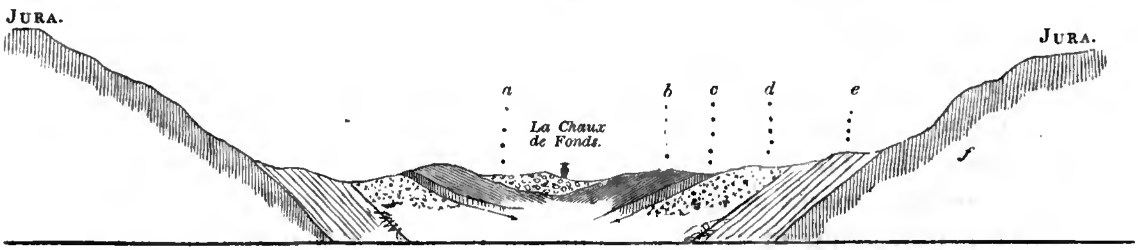


Immediately on leaving the town of Neuchâtel the road begins to rise, although the passage across the first range of the Jura is rendered more easy by its following the course of a transverse valley, which brings a mountain torrent from the first and most Easterly valley to the lake of Neuchâtel. The naked walls of rock exhibited on each side of the road show clear marks of the violent dislocation which must have accompanied the upheaving of the mountain chain, and we can trace easily the direction of the anticlinal axis, and the contortions of the strata near the highest part of the range. The descent on the Western side is rapid but not very long, and brings us quickly to the little town of Vallengy, which is built upon a considerable bed of gravel, the superstratum of a valley, without doubt, of tertiary formation. The valley thus covered up is a fair specimen of many of those occurring between the two most Easterly parallel ranges of the Jura: they are for the most part desolate and barren, now and then watered by a small stream, but then only presenting a little pleasing scenery close to the water's edge. I should imagine that they had been formed rather by the action of submarine currents depositing gravel, than by any regular subsidence of transported matter in a lake.

Crossing this valley, whose breadth is here nearly two miles, we come again to the secondary rocks of the Jura, and the road passes over the middle and highest range of those mountains. At a very high elevation, and enclosed between two ridges of nearly 4500 feet, occurs a second small valley, much more narrow and insignificant than the one before mentioned; and after having crossed it, there is a sudden and rapid descent, leading down to the third and principal valley, that of la Chaux de Fonds, the examination of which was the main object of my excursion. In physical features, as well as geological structure, this valley has all the character of a lacustrine deposit, left dry, either by the silting up of a mountain lake, or gradual evaporation for want of a sufficient supply of water. There is certainly no outlet for water, and scarcely a single running stream in its whole extent.

The village of la Chaux de Fonds is near the northern extremity of the valley, and about midway between the mountains on the east and west. It is built partly upon a small bed of clay and marl, marked (*a*) in Section 3,

SECTION III. ACROSS THE VALLEY OF LA CHAUX DE FONDS.



Level of the Lake of Neuchâtel. *Scale 2 inches to a mile.*

and partly upon a fresh-water limestone, the upper beds of which alternate with the marls above. On each side of this band of limestone, marked (*b*), there comes out another series of marls (*c*), resting upon the Molasse (*d*) which is here of no great thickness, and overlies a portion (probably the lower part) of the chalk formation (*e*), immediately below which in this part of the district are the upper oolite beds of the Jura (*f*).

In the uppermost of all these beds, resting on the fresh-water limestone, there have been discovered, in digging foundations for houses, several fragments of bones, among which were teeth in tolerable preservation. These bones, being examined by competent anatomists*, have been referred to the following genera :—Anoplotherium, Palœotherium, and Lophiodon, Hippopotamus, Camelopardalis, Equus, Deinotherium, Elephas, and Rhinoceros. To the bed containing these fossils, and the circumstances under which they occur, I am desirous now of directing attention.

The bed I have already sufficiently described as a black earthy deposit, alternating with calcareous bands. It is pretty regularly stratified, and I was struck with the probability there seemed of its having been formed while the lake, which doubtless once covered the whole valley, was so far dried up as to resemble a marshy pond, in which the bones would be preserved as in a peat bog. Of the species determined, I believe five have been identified as occurring also in the Paris Basin; the others would seem to belong to a more recent period, and perhaps we should rather refer to the tertiary beds of Bordeaux, and the valleys of the Garonne and Loire, than to the neighbourhood of Paris for analogies. The Miocene period of Mr Lyell has already been suggested by that gentleman as the probable date of the Jura tertiaries, and the discovery of these fossils would tend to confirm his opinion.

The Molasse, however, being the substratum, and resting immediately upon the cretaceous beds, it is clearly an older deposit, perhaps existing as the bottom of an ancient sea, before the disturbances and elevations, which formed the valleys of the Jura, and raised them to their present position, took place.

* Most of the specimens were determined by Professor Agassiz, and many of them sent to Paris to be compared with the fossils examined and named by Cuvier, and found in the Lower Tertiary formation of the Calcaire grossière.

In conclusion, the Tertiary Geology of the South-West of Switzerland may be said to be separable under three heads; first, the great deposit of Molasse, which appears, from all we can tell, to be of marine origin; secondly, the fresh-water marls and lignite bands occurring in the Molasse, but very local, and apparently near the upper part; and thirdly, the overlying beds of marl and limestone in the valleys of the Jura, which alone can be compared with the better developed systems in other parts of Europe; but since, from the general dip of the sandstone, that portion of it in the Jura valleys would seem to have been the earliest formed, there is no reason why the overlying beds there should be very much newer than the lignite near the Alps. The period therefore to which the Molasse must be referred, still remains in doubt. It also results from the dips and observations recorded, that the so called great Helvetic Basin is in fact no basin at all, but a vast accumulation of sandstone, formed probably upon an inclined plane, and then tilted to a greater or less angle into its present position. The smaller valleys are indeed true basins, but the structure of many of them, especially the most Easterly, is a point, I think, yet to be determined.

It is obvious that much remains to be done in determining the true geological relations of the Molasse, its fossils, and the varieties of its dip; and I would especially direct attention to the limestone near the Southern extremity of the lake of Neuchâtel, which is the most promising of any part of this tiresome formation. Should there be found here any fossils, they must possess great interest; and I regretted extremely the want of opportunity which prevented me from examining accurately the whole neighbourhood. It is very accessible, being close to the high road between Yverdon and Moudon, and certainly deserves the attention of any geologist travelling in that part of Switzerland.

Should anything I have said lead to the determination of this and other points in Swiss Tertiary Geology, my object in bringing this paper before the Society will be fully accomplished.

D. T. ANSTED.

Jesus College,
March 1841.

IX. *On the Quantity of Light intercepted by a Grating placed before a Lens; and on the Effect produced by Interference. By the Rev.* PHILIP KELLAND, M.A., F.R.SS.L. & E. *late Fellow of Queens' College, Cambridge; Professor of Mathematics in the University of Edinburgh.*

[Read *March* 30, 1840.]

FROM the remarkable appearances presented by the interception of a part of the light proceeding from a small luminous body towards the object-glass of a telescope, it may very naturally be supposed that vibrations are suffered to exist, which would otherwise be destroyed by interference; and that consequently a less quantity of light is stopped by the grating than that which is actually incident on it. That light actually appears from the application of the grating, where there would be little or none without it, is most certain; and that this circumstance arises from the want of interference, alluded to above, there can be no doubt. Should we then expect, notwithstanding the cause to which we attribute the phenomenon, (not to speak of the phenomenon itself) to find exactly the same quantity of light on the other side the lens, or at least in its field of view, as would correspond to the spaces left open by the grating?

When first asked my opinion on this subject, I had no hesitation in pronouncing that, previous to calculation, I should expect to find *more* light transmitted through a grating, than in proportion to the space left uncovered. My idea of the matter was this: certain vibrations are not destroyed when the grating is applied, which would be destroyed in the contrary case; whereas there is nothing to affect those spaces from which vibrations are excluded, so as to render this nugatory. This reasoning, subsequent consideration convinces me is incorrect. It appears that, although the stoppage of vibrations by the wires does bring into operation that which must otherwise have been destroyed, yet the same stoppage causes to disappear certain of the vibrations corresponding with the uncovered part, which would, in the contrary case, appear in the aggregate of all the motion.

My attention was called to this subject by Professor Forbes, who has been prosecuting an experimental enquiry into the effect produced by screens on the transmission of radiant heat. The curious fact which he has established relative to the difference in amount of the stoppage produced in light and dark heat,—or at least in two different kinds of heat, which he has found to be operated on very differently in other matters—promises to give us an insight into the characteristic properties of light and heat, provided it appear that one kind of heat is, in the case before us, acted on in the same manner as light is, in like circumstances. But perhaps it is too much to hope that we shall distinguish betwixt *light* and heat, uncertain as we are of the intensity of the former, by which its nature might be contrasted with that of the latter. It may then be expected, rather, that we shall be put in the way of distinguishing between heat and undulations; distinction being, if I mistake not, absolutely necessary, as well as obviously pointed at, by the very experiments which seem most strongly to identify the two with each other.

I forbear, however, entering on this subject at present, although I am deeply interested in it, as well on account of its intrinsic importance, as of its bearing on my own views of the Theory of Heat. I shall therefore, without further preface, proceed to the question in hand.

Our Problem is this:—

A series of equal parallelograms are placed before a lens, to find the whole quantity of light received on a screen, placed perpendicular to the axis of the lens at its focus.

The solution of the Problem for finding the intensity at any one individual point will be found in Airy's Tracts, p. 328, at the foot of Art. 83.

The expression is this:

$$4e^2f^2\left(\frac{\lambda b}{2\pi q f}\sin\frac{2\pi q f}{\lambda b}\right)^2 \cdot \left(\frac{\lambda b}{\pi p e}\sin\frac{\pi p e}{\lambda b}\right)^2 \cdot \left(\frac{\sin\frac{p(e+g)\pi}{\lambda b}m}{\sin\frac{p(e+g)\pi}{\lambda b}}\right)^2.$$

The notation is as follows:

e is the breadth of one of the openings between the wires;

g the breadth of a wire;

p, q the co-ordinates of the point, measured along the screen from the focus of the lens,

p being perpendicular to the wires;

m the number of openings and of wires.

To obtain the whole quantity of light, then, we must multiply this expression by $4dp\,dq$, and integrate between the limits 0 and ∞.

Let the result of the integration for q give

$$\frac{\pi H e^2}{\lambda b}\int_0^\infty dp\left(\frac{\sin x}{x}\right)^2\left(\frac{\sin\left(1+\frac{g}{e}\right)mx}{\sin\left(1+\frac{g}{e}\right)x}\right)^2 \text{ by writing } x \text{ for } \frac{\pi p e}{\lambda b}.$$

Then $dp = \frac{\lambda b}{\pi e}dx$, so that the expression becomes

$$He\int_0^\infty dx\left(\frac{\sin x}{x}\right)^2\left(\frac{\sin rmx}{\sin rx}\right)^2 \text{ if } r = 1+\frac{g}{e}.$$

1. Now first, we must integrate this expression on the hypothesis that the aperture is *uninterrupted*, or that $m = 1$. I shall make use of the well known formula of LAPLACE: viz.

$$\int_0^\infty \frac{a\cos qx\,.\,dx}{a^2+x^2} = \frac{\pi}{2}e^{-aq}$$

the particular value of a being 0.

$$\text{Thus } \int_0^\infty \frac{dx}{x^2} = \frac{\pi}{2a}$$

$$\int_0^\infty .\,\frac{\cos x dx}{x^2} = \frac{\pi}{2a}e^{-a}$$

$$\int_0^\infty .\,\frac{\cos 2x dx}{x^2} = \frac{\pi}{2a}e^{-2a}$$

$$\&c. = \&c.$$

Supplying these values, we obtain,

$$\int_0^\infty dx\left(\frac{\sin x}{x}\right)^2 = \frac{1}{2}\int_0^\infty dx\frac{1-\cos 2x}{x^2}$$

$$= \frac{\pi}{4a}(1-e^{-2a}), \ a \text{ being } \textit{always} = 0$$

$$= \frac{\pi}{2};$$

∴ the intensity on this supposition is $He\frac{\pi}{2}$, or as we will write it, to denote that e stands now for the semi-aperture, $HE\frac{\pi}{2}$.

We find then, that the whole quantity of light incident is exactly that which corresponds to the open space between the bars, no effect being produced by interference, or its destruction.

2. Next, as a particular case, in order to make the process intelligible, we will find the illumination when the breadths of the wires are equal to the openings between them.

If, for instance, there be two wires, the general formula gives

$$He\int_0^\infty \left(\frac{\sin x}{x}\right)^2\left(\frac{\sin 4x}{\sin 2x}\right)^2 dx$$

$$= He\int_0^\infty \left(\frac{\sin x}{x}\right)^2 4\cos^2 2x\, dx$$

$$= 4He\int_0^\infty \frac{1}{x^2}\left(\sin^2 x - 4\sin^4 x + 4\sin^6 x\right)$$

$$= 4He\int_0^\infty \frac{1}{x^2}\left(\frac{1}{4} - \frac{3}{8}\cos 2x + \frac{1}{4}\cos 4x - \frac{1}{8}\cos 6x\right)$$

$$= 4He\frac{\pi}{2a}\left(\frac{1}{4} - \frac{3}{8}e^{-2a} + \frac{1}{4}e^{-4a} - \frac{1}{8}e^{-6a}\right)$$

$$= 2\pi He\left(\frac{3}{4} - 1 + \frac{6}{8}\right)$$

$$= \pi He.$$

But $e = \frac{1}{4}E$; since the original aperture is divided into four equal parts, two of which are appropriated to the openings, and two to the wires.

$\therefore$ The intensity $= \frac{\pi}{4}HE$

$= \frac{1}{2}$ the result found for the intensity when there are no wires.

3. Thirdly, let us retain the hypothesis, that the breadths of the wires are the same as the openings between them, but suppose the number of wires and of openings to be any number whatever, (m).

$$\text{Write } \frac{\theta - \theta^{-1}}{2\sqrt{-1}} \text{ for } \sin x:$$

then the expression $(\sin x)^2 \left(\frac{\sin 2mx}{\sin 2x}\right)^2$ is put under the form,

$$-\frac{1}{4}(\theta - \theta^{-1})^2 \left(\frac{\theta^{2m} - \theta^{-2m}}{\theta^2 - \theta^{-2}}\right)^2,$$

$$\text{or } -\frac{1}{4}\left(\frac{\theta^{2m} - \theta^{-2m}}{\theta + \theta^{-1}}\right)^2.$$

$$\text{Let } \left(\frac{\theta^{2m} - \theta^{-2m}}{\theta + \theta^{-1}}\right)^2 = a_0\theta^{4m-2} + a_1\theta^{4m-4} + \&c. + a_{4m-2}\theta^{-(4m-2)};$$

$$\therefore \theta^{4m} - 2 + \theta^{-4m} = a_0\theta^{4m} + a_1\theta^{4m-2} + a_2\theta^{4m-4} + \ldots + a_{4m-2}\theta^{-(4m-4)}$$
$$+ 2a_0\theta^{4m-2} + 2a_1\theta^{4m-4} + \ldots + 2a_{4m-2}\theta^{-(4m-2)}$$
$$+ a_0\theta^{4m-4} + \ldots + a_{4m-2}\theta^{-4m};$$

from which we obtain by equating coefficients,

$$1 = a_0,$$
$$0 = a_1 + 2a_0,$$
$$0 = a_2 + 2a_1 + a_0,$$
$$0 = a_3 + 2a_2 + a_1,$$
$$\ldots = \ldots\ldots\ldots\ldots$$

$$-2 = a_{2m} + 2a_{2m-1} + a_{2m-2},$$
$$0 = a_{2m+1} + 2a_{2m} + a_{2m-1},$$
$$\ldots = \ldots\ldots\ldots\ldots$$
$$0 = 2a_{4m-2} + a_{4m-3},$$
$$a_{4m-2} = 1.$$

These results give

$$a_0 = 1, \quad a_1 = -2, \quad a_2 = 3, \quad a_3 = -4, \text{ \&c.}$$

$$a_{2m-1} = -2m, \quad a_{2m} = -2 + 4m - 2m + 1 = 2m - 1,$$

$$a_{2m+1} = -(2m-2) \text{ \&c. } a_{4m-3} = -2, \quad a_{4m-2} = 1.$$

By substitution, therefore, the general expression

$$He\int_0^\infty dx \left(\frac{\sin x}{x}\right)^2 . \left(\frac{\sin rmx}{\sin rx}\right)^2 \text{ becomes}$$

$$-\tfrac{1}{4} He\int_0^\infty \frac{dx}{x^2} 2 \{\cos (4m-2)x - 2\cos (4m-4)x$$

$$+ 3\cos (4m-6)x - \text{\&c.} \ldots\ldots + (2m-1)\cos 2x - m\}$$

or $-\dfrac{\pi}{4}\dfrac{He}{a}\{e^{-(4m-2)a} - 2e^{-(4m-4)a} + 3e^{-(4m-6)a} - \text{\&c.} + (2m-1)e^{-2a} - m\}$

$$\text{But } 1 - 2 + 3 - \text{\&c.} + (2m-1) - m = 0\,;$$

∴ the expression gives ; observing that $a = 0$,

$$+\frac{\pi}{4} He(\overline{4m-2} - 2\,\overline{4m-4} + 3\,\overline{4m-6} - \text{\&c.} + \overline{2m-1}\,.\,2).$$

Now $4m - 2 - 2(4m-4) + 3(4m-6) - \text{\&c.} + (2m-1)\,.\,2$

$$= 4m\{1 - 2 + 3 - \text{\&c.} + (2m-1)\}$$
$$- 2\{1^2 - 2^2 + 3^2 - \text{\&c.} + (2m-1)^2\}$$
$$= 4m^2 - 2m(2m-1) = 2m.$$

Hence the expression becomes

$$\frac{\pi}{2} Hem:$$

that is, $\frac{\pi}{2} H \times$ space left uncovered.

Or, which is the same thing, since $m \,.\, 2e = E$,

$$\text{the expression is } \frac{1}{2} \cdot \frac{\pi}{2} HE,$$

or, one half the quantity of light which would fall were there no grating.

4. To give one more particular case, we will take that in which $g = 2e$, or the breadth of the wire is double that of the opening between the wires.

Our expression then becomes $He \int_0^\infty \left(\frac{\sin x}{x}\right)^2 \left(\frac{\sin 3mx}{\sin 3x}\right)^2 dx,$

$$\text{and } -4(\sin x)^2 \left(\frac{\sin 3mx}{\sin 3x}\right)^2 = (\theta - \theta^{-1})^2 \left(\frac{\theta^{3m} - \theta^{-3m}}{\theta^3 - \theta^{-3}}\right)^2 = \left(\frac{\theta^{3m} - \theta^{-3m}}{\theta^2 + 1 + \theta^{-2}}\right)^2.$$

Assume this expression to be expanded in the form

$$a_1 \theta^{6m-4} + a_2 \theta^{6m-6} + \ldots\ldots$$

$$+ a_2 \theta^{-(6m-6)} + a_1 \theta^{-(6m-4)}:$$

the terms from the beginning and end of the series having obviously equal coefficients,

$$\therefore \theta^{6m} - 2 + \theta^{-6m} = (\theta^4 + 2\theta^2 + 3 + 2\theta^{-2} + \theta^{-4})$$

$$\times \left\{a_1 \theta^{6m-4} + a_2 \theta^{6m-6} + \ldots\ldots + a_2 \theta^{-(6m-6)} + a_1 \theta^{-(6m-4)}\right\}$$

$$\begin{aligned} = a_1\theta^{6m} + a_2\theta^{6m-2} + a_3\theta^{6m-4} &+ a_4\theta^{6m-6} + a_5\theta^{6m-8} + \ldots \\ + 2a_1\theta^{6m-2} + 2a_2\theta^{6m-4} &+ 2a_3\theta^{6m-6} + 2a_4\theta^{6m-8} + \ldots \\ + 3a_1\theta^{6m-4} &+ 3a_2\theta^{6m-6} + 3a_3\theta^{6m-8} + \ldots \\ &+ 2a_1\theta^{6m-6} + 2a_2\theta^{6m-8} + \ldots \\ &+ a_1\theta^{6m-8} + \ldots \end{aligned}$$

Equating coefficients, we obtain

$$a_1 = 1,$$

$$a_2 + 2a_1 = 0$$

$$a_3 + 2a_2 + 3a_1 = 0$$

$$a_4 + 2a_3 + 3a_2 + 2a_1 = 0$$

$$a_5 + 2a_4 + 3a_3 + 2a_2 + a_1 = 0$$

$$a_6 + 2a_5 + 3a_4 + 2a_3 + a_2 = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots$$

$$a_{3m+1} + 2a_{3m} + 3a_{3m-1} + 2a_{3m-2} + a_{3m-3} = -2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0;$$

$$\therefore a_1 = 1,\ a_2 = -2$$

$$a_3 = 1,\ a_4 = 2,\ a_5 = -4$$

$$a_6 = 2,\ a_7 = 3,\ a_8 = -6$$

$$a_9 = 3,\ a_{10} = 4,\ a_{11} = -8$$

$$\ldots = \ldots,\ \ldots = \ldots,\ \ldots = \ldots\ldots$$

Suppose this law to hold true for any three consecutive terms;

$$\text{that is, let } \left.\begin{array}{l} a_{3n} = n, \\ a_{3n} + 1 = n + 1, \\ a_{3n} + 2 = -2n - 2; \end{array}\right\}$$

$$\therefore a_{3n+3} = 4n + 4 - 3n - 3 - 2n + 2n$$

$$= n + 1,$$

$$a_{3n+4} = -2n - 2 + 6n + 6 - 2n - 2 - n$$

$$= n + 2,$$

$$a_{3n+5} = -2n - 4 - 3n - 3 + 4n + 4 - n - 1$$

$$= -2n - 4:$$

hence the law holds good for the next greater consecutive terms, and is consequently general.

Also the middle term is

$$a_{3m-1} = 2m.$$

We thus obtain as the value of the intensity,

$$-\frac{He}{2}\int_0^\infty dx \frac{1}{x^2}\{\cos(6m-4)x - 2\cos(6m-6)x + \ldots + m\}$$

$$= -\frac{\pi He}{4a}\{e^{-(6m-4)a} - 2e^{-(6m-6)a} + \ldots + m\}$$

$$= -\frac{\pi}{4}\frac{He}{a}\{1 - 2 + 1 + 2 - 4 + \ldots \text{ to } (3m-2) \text{ terms} + m\}$$

$$+ \frac{\pi He}{4}\{1.(6m-4) - 2(6m-6) + \ldots \text{ to } (3m-2) \text{ terms}\}.$$

Now $1 - 2 + 1 + \ldots + m$ is obviously half the above expansion, when $\theta = 1$, and is consequently zero.

$$\text{Also } 1.(6m-4) - 2(6m-6) + \ldots\ldots$$

$$= 1(6m-4) + 2(6m-10) + \ldots\ldots + m(6m - \overline{6m-2})$$

$$- 2(6m-6) - 4(6m-12) - \ldots\ldots - 2(m-1)(6m - \overline{6m-6})$$

$$+ 1(6m-8) + 2(6m-14) + \ldots\ldots + (m-1)(6m - \overline{6m-4})$$

$$= 1.(12m-12) + 2(12m-24) + \ldots + (m-1)(12m - \overline{12m-12})$$

$$- 2(6m-6) - 4(6m-12) - \ldots\ldots - 2(m-1)(6m - \overline{6m-6})$$

$$+ m(6m - \overline{6m-2})$$

$$= 2m.$$

Hence the whole intensity is $\frac{\pi}{2} Hem = \frac{\pi}{2} H \times$ space left uncovered, the same result as before.

5. Lastly, let us take the most general case, of a grating in which the thickness of the bars bears any proportion whatever to the spaces left uncovered.

Adopting the general expression, we have now to find the value of

$$(\sin x)^2 \left(\frac{\sin rmx}{\sin rx}\right)^2 \text{ or } -\frac{(\theta-\theta^{-1})^2}{4}\cdot\left(\frac{\theta^{rm}-\theta^{-rm}}{\theta^r-\theta^{-r}}\right)^2.$$

$$\text{Assume } \left(\frac{\theta^{rm}-\theta^{-rm}}{\theta^r-\theta^{-r}}\right)^2 = \theta^{2r(m-1)} + a_2\theta^{2r(m-2)} + \ldots\ldots + \theta^{-2r(m-1)},$$

$$\begin{aligned}\therefore\ \theta^{2rm} - 2 + \theta^{-2rm} = \theta^{2rm} &+ a_2\theta^{2r(m-1)} + a_3\theta^{2r(m-2)} + \ldots \\ &- 2\theta^{2r(m-1)} - 2a_2\theta^{2r(m-2)} - \ldots \\ &+ \theta^{2r(m-2)} + \ldots\end{aligned}$$

which by equating coefficients, gives

$$\begin{aligned} a_2 - 2 &= 0, \\ a_3 - 2a_2 + 1 &= 0, \\ a_4 - 2a_3 + a_2 &= 0, \\ \ldots\ldots\ldots\ldots\ldots &= \ldots \\ a_{m+1} - 2a_m + a_{m-1} &= -2, \\ \ldots\ldots\ldots\ldots\ldots &= 0: \end{aligned}$$

$$\text{hence } a_2 = 2, \quad a_3 = 3, \quad a_4 = 4, \text{ \&c.}$$

$$\begin{aligned}\text{and } a_{m+1} &= 2m - (m-1) - 2 \\ &= m - 1 \\ \ldots\ldots &= \ldots\ldots\end{aligned}$$

that is, the coefficients form an arithmetic series, increasing up to m, and then diminishing down to 1.

By multiplying by the factor $(\theta - \theta^{-1})^2$, we obtain

$$\begin{aligned}&\theta^{2r(m-1)+2} + 2\theta^{2r(m-2)+2} + \ldots + m\theta^2 + \ldots + \theta^{-2r(m-1)+2} \\ &+ \theta^{-(2r\overline{m-1}+2)} + 2\theta^{-(2r\overline{m-2}+2)} + \ldots + m\theta^{-2} + \ldots + \theta^{2r(m-1)-2} \\ &- 2\left\{\theta^{2r(m-1)} + 2\theta^{2r(m-2)} + \ldots + m + \ldots + \theta^{-2r(m-1)}\right\} \\ &= 2\left\{\cos(2r\overline{m-1}+2)x + 2\cos(2r\overline{m-2}+2)x + \ldots\right. \\ &+ m\cos 2x + (m-1)\cos(2r-2)x + \ldots + \left.\cos[2r(m-1)-2]x\right\} \\ &- 4\left\{\cos 2r(m-1)x + 2\cos 2r(m-2)x + \ldots + (m-1)\cos 2rx\right\} \\ &- 2m = K.\end{aligned}$$

Hence the intensity is

$$-\frac{He}{4}\int_0^\infty \frac{dx}{x^2} K = -\frac{\pi He}{4a}\{e^{-(2r\overline{m-1}+2)a} + 2e^{-(2r\overline{m-2}+2)a} + \ldots + me^{-2a}$$

$$+ (m-1)e^{-(2r-2)a} + (m-2)e^{-(4r-2)a} + \ldots + e^{-(2r\overline{m-1}-2)a}$$

$$-2\left[e^{-2r(m-1)a} + 2e^{-2r(m-2)a} + \ldots + (m-1)e^{-2ra}\right]$$

$$-m\}$$

$$= -\frac{\pi He}{4a}.\{1 + 2 + \ldots\ldots + m + (m-1) + \ldots + 1$$

$$-2(1 + 2 + \ldots + \overline{m-1}) - m\}$$

$$+\frac{\pi He}{4}.\{1.(2r.\overline{m-1} + 2) + 2(2r\overline{m-2} + 2) + 3(2r\overline{m-3} + 2)$$

$$+ \ldots + (m-1)(2r+2) + m.2$$

$$+(m-1)(2r-2) + (m-2)(4r-2) + \ldots + 2r(m-1) - 2$$

$$-2\left[2r.\overline{m-1} + 2.2r(m-2) + \ldots + (m-1)2r\right]\}$$

$$= \frac{\pi He}{4}.\{4r(m-1) + 2.4r(m-2) + \ldots + (m-1)4r$$

$$+ 2m - 4r(m-1) - 2.4r(m-2) - \ldots - (m-1)4r\}$$

$$= \frac{\pi Hem}{2} = \frac{\pi H}{2} \times \text{space left uncovered.}$$

Thus it appears that the whole quantity of light is not at all affected by the diminution of interference. For we obtain, *whole quantity of light on the screen* : *that which falls on the object-glass* :: *area of the uncovered part of the glass* : *whole area of the glass.*

It is unnecessary to dwell on this result. That it is a strong confirmation of the undulatory theory, as far as regards two hypotheses respecting the *intensity*, and the *vibrations in different directions*, cannot be doubted.

The common assumption, that the intensity is measured by the square of the excursion of a vibrating particle, although bearing a great air of probability, is still not so obvious as to derive no benefit from a confirmation such as our conclusions tend to give it.

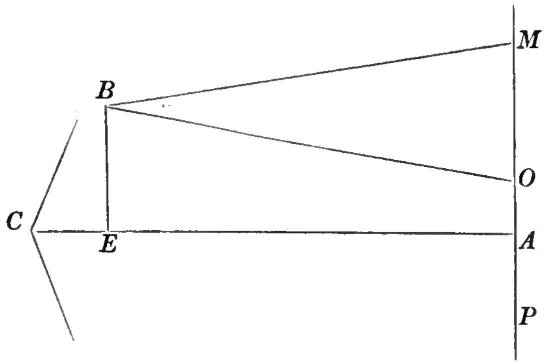
The hypothesis respecting the intensity of vibrations in different *directions*, and at different distances, as stated by Mr Airy, is this: *that a vibrating particle transmits vibrations equally in all directions, but with an intensity varying inversely as the distance.* This hypothesis is not altogether conformable to our conclusion, which appears to require that vibrations transmit forces equally in all directions, and to all distances. Fortunately, none of the *approximate* results deduced from either hypothesis are vitiated by it, since the variation of distance is not taken into consideration in the solution. Of course, these observations are based on the supposition, that the division of a complete wave into elementary portions, in the manner always employed to effect the exhibition of results deducible from a change of circumstances in the mode of transmission, is allowable. My object, at present, being rather the demonstration of a property of undulations, than an application to the theory either of light or heat, I have contented myself with alluding to the bearings of the result to which we have arrived. What has been said will be confirmed by the following problem, with which the preceding is intimately connected.

"The whole intensity of light reflected at the surfaces of two plane mirrors, inclined to each other at any angle, is not altered by the interference of the light from the one mirror with that reflected from the other."

To this problem we shall annex the same limitations, and apply the same processes as to that already solved. That is to say, we shall conceive a lens placed before the mirrors, so as to bring the reflected light to two foci lying in a line perpendicular to that which bisects the angle between the mirrors.

Let C be the projection of the line of intersection of the mirrors; O, P, the foci to which the rays from the mirrors respectively converge. Then each wave on leaving the lens will be a portion of a sphere, of which the centre is the point of convergence.

Let b be the radius of the sphere $= BO$, $AO = f$, $AM = p$, $AE = x$,



$EB = y + f$, BE being perpendicular to AC. Then the vibration at the point M due to a vibration C at B is

$$c \sin \frac{2\pi}{\lambda} (vt - BM).$$

$$\text{But } BM^2 = (p - f - y)^2 + x^2$$

$$= (p - f)^2 - 2(p - f)y + x^2 + y^2$$

$$= (p - f)^2 - 2(p - f)y + b^2;$$

$$\therefore\ BM = B - \frac{p - f}{b} . y \text{ nearly.}$$

Hence the vibration at M produced by the upper mirror, as far as its projection on the plane of the paper is concerned, is

$$\Sigma c\delta y \sin \frac{2\pi}{\lambda} \{vt - (p - f^2) - b^2 + 2(p - f') - y\}.$$

Also, if B be not in the plane of the paper, BM^2 becomes

$$(p - f - y)^2 + x^2 + z^2, \text{ or } (p - f)^2 - 2(p - f)y + b^2,$$

as before.

Hence, the expression for the vibration is

$$c\int_{-f}^{g} dy \sin\frac{2\pi}{\lambda}\left(vt - B + \frac{p-f}{b}y\right)$$

$$= \frac{c\lambda b}{2\pi(p-f)}\left\{\cos\frac{2\pi}{\lambda}\left(vt - B - \frac{p-f}{b}f\right)\right.$$

$$\left. - \cos\frac{2\pi}{\lambda}\left(vt - B + \frac{(p-f)g}{b}\right)\right\}$$

$$= \frac{c\lambda b}{\pi(p-f)}\sin\frac{\pi}{\lambda}\frac{(p-f)(g+f)}{b}\sin\frac{2\pi}{\lambda}\left(vt - B + \frac{(p-f)(g-f)}{2b}\right).$$

In the same way, the vibration due to the second mirror is

$$c\int_{-f}^{g} dy \sin\frac{2\pi}{\lambda}\left(vt - B' - \frac{(p+f)y}{b}\right)$$

$$= \frac{c\lambda b}{2\pi(p+f)}\left\{\cos\frac{2\pi}{\lambda}\left(vt - B' + \frac{(p+f)f}{b}\right)\right.$$

$$\left. - \cos\frac{2\pi}{\lambda}\left(vt - B' - \frac{(p+f)g}{b}\right)\right\}$$

$$= \frac{c\lambda b}{\pi(p+f)}\sin\frac{\pi}{\lambda}\frac{(p+f)(g+f)}{b}\sin\frac{2\pi}{\lambda}\left(vt - B' - \frac{(p+f)(g-f)}{2b}\right)$$

B' being equal to $B + \frac{2fp}{b}$:

so that the whole vibration at the point M is

$$M\sin\frac{2\pi}{\lambda}\left(vt - B + \frac{(p-f)(g-f)}{2b}\right)$$

$$+ N\sin\frac{2\pi}{\lambda}\left(vt - B' - \frac{(p+f)(g-f)}{2b}\right);$$

where M and N are the coefficients of vibration due to each mirror respectively.

If we expand these in terms of the common circular arc, which is

$$\frac{2\pi}{\lambda}\left(vt - B - \frac{f(g-f)}{2b}\right),$$

we obtain, calling this arc θ for the sake of brevity,

$$\left\{M\cos\frac{\pi}{\lambda}.\frac{p(g-f)}{b} + N\cos\frac{\pi}{\lambda}\frac{p(g-f)+4pf}{b}\right\}\sin\theta,$$

$$+\left\{M\sin\frac{\pi}{\lambda}\frac{p(g-f)}{b} - N\sin\frac{\pi}{\lambda}\frac{p(g-f)+4pf}{b}\right\}\cos\theta;$$

so that the intensity at the point is represented by

$$M^2 + N^2 + 2MN\cos\frac{2\pi}{\lambda}.\frac{p(g+f)}{b}.$$

We must next integrate this expression between the limits $-\infty$ and $+\infty$ for p.

$$\text{Now } \int_{-\infty}^{\infty} M^2 dp = \int_{-\infty}^{\infty}\frac{c^2\lambda^2 b^2}{\pi^2(p-f)^2}\sin^2\frac{\pi}{\lambda}\frac{(p-f)(g+f)}{b}.d(p-f)$$

$$= \frac{c^2\lambda^2 b^2}{\pi^2}\int_0^{\infty}\left(1-\cos\frac{2\pi}{\lambda}.\frac{(p-f)(g+f)}{b}\right)\frac{d.(p-f)}{(p-f)^2}$$

$$= \frac{c^2\lambda^2 b^2}{\pi^2}\frac{\pi}{2a}\left(1-e^{-\frac{2\pi}{\lambda}.\frac{(g+f)}{b}.a}\right), \text{ } a \text{ being equal to zero,}$$

$$= \frac{c^2\lambda^2 b^2}{\pi^2}.\frac{\pi^2}{\lambda}.\frac{g+f}{b},$$

$$= c^2.\lambda b(g+f).$$

$$\text{Similarly, } \int_{-\infty}^{\infty} N^2 dp = c^2\lambda b(g+f).$$

Hence we find that the whole effect of each mirror is proportional to its aperture: which result is strongly confirmatory of the general character of our calculations.

$$\text{Lastly, } MN\cos\frac{2\pi}{\lambda}\frac{p(g+f)}{b}$$

$$= \frac{c^2\lambda^2 b^2}{\pi^2(p^2-f^2}\sin\frac{\pi}{\lambda}\frac{(p-f)(g+f)}{b}\sin\frac{\pi}{\lambda}\frac{(p+f)(g+f)}{b}\cos\frac{2\pi}{\lambda}\frac{p(g+f)}{b},$$

$$= \frac{c^2\lambda^2 b^2}{2\pi^2(p^2-f^2)}\left\{\cos\frac{2\pi}{\lambda}\frac{f(g+f)}{b} - \cos\frac{2\pi}{\lambda}\frac{p(g+f)}{b}\right\}\cos\frac{2\pi}{\lambda}\frac{p(g+f)}{b},$$

Now the circular functions in this expression are

$$(\cos af - \cos ap)\cos ap, \text{ calling } \frac{2\pi}{\lambda}\frac{g+f}{b},\ a.$$

But by Laplace's Formula,

$$\int_0^\infty \frac{\cos a\,p}{p^2 - f^2} = \frac{\pi}{2f\sqrt{-1}}\,e^{-af\sqrt{-1}};$$

and the integral between our limits is merely the double of this;

$\therefore$ the integral of the term

$$MN\cos ap = A\left\{\cos af e^{-af\sqrt{-1}} - \tfrac{1}{2} - \tfrac{1}{2}e^{-2af\sqrt{-1}}\right\}$$

$$= A\left\{\cos af\,\overline{\cos af - \sqrt{-1}\sin af} - \tfrac{1}{2} - \tfrac{1}{2}(\cos 2af - \sqrt{-1}\sin 2af)\right\}$$

$$= A\left\{\cos^2 af - \tfrac{1}{2} - \tfrac{1}{2}\cos 2af - \sqrt{-1}\,(\sin af\cos af - \tfrac{1}{2}\sin 2af)\right\}$$

$$= 0$$

a very remarkable result.

If M be not in the plane COP, there is a factor $\dfrac{\sin qh}{q}$ in M and N, which amounts to the factor $\displaystyle\int_0^\infty \left(\frac{\sin qh}{q}\right)^2 dq$ or $\dfrac{\pi h}{2}$ in the final result.

We find then that the whole intensity is the sum of the intensities due to each of the mirrors separately. Should the form of the function M^2 as integrated for the whole space be objected to, the only reply is, that one or other of two things must be supposed; either 1° that the integration for spaces perpendicular to the plane of the paper would take away λ, or 2° that the intensity is a function of the length of the wave. In either case our conclusion is correct. There is evidently *some* factor required to render this result of the same dimensions as that with which we set out. Perhaps I am not warranted in assuming, from the coincidence of my results with the principle of *vis viva*, and their consequent probability, that this factor is not variable from point to point. When the question first arose in my mind respecting this matter, I thought to answer it at once by an appeal to the transformations effected by the "Differential Calculus to *any* indices." Although the result of this appeal is very far from satisfactory, I do not think it will be deemed an unpardonable digression to take it here.

The principle assumed as the basis of calulation is this*: "The effect of any wave in disturbing any given point, may be found by taking the front of the wave at any given time, dividing it into an indefinite number of small parts, considering the agitation of each of these small parts as the cause of a small wave, which will disturb the given point, and finding, by summation or integration, the aggregate of all the disturbances of the given point, produced by the small waves coming from all points of the great wave."

I took, then, the simplest case which can be conceived, viz. that of an infinite plane wave. There can be no doubt that the result in this case ought to be the following: that the disturbance produced is the same in intensity as that corresponding to one of the points in the disturbing wave.

Let b be the perpendicular distance of the given point from the wave, then it is evident that if $c \sin \frac{2\pi}{\lambda}(vt - x)$ be the disturbance of any point in the wave, the effect produced, according to the above principle, will be represented by

$$2\pi \int_0^\infty c r dr \sin \frac{2\pi}{\lambda}\left(vt - \sqrt{r^2 + b^2}\right) \times \text{ some quantity.}$$

Nor is it less evident that the result actually is $c \sin \frac{2\pi}{\lambda}(vt - b)$.

What therefore is the multiplier in question? If it is not constant, it must be some function of r.

Denote r^2 by z, and let the multiplier be $f(z)$:

then our equation assumes the form

$$\pi \cdot \int_0^\infty dz \sin \frac{2\pi}{\lambda}\left(vt - \sqrt{z + b^2}\right) f(z) = \sin \frac{2\pi}{\lambda}(vt - b).$$

But by the very elegant theorem of M. Liouville†

$$\int_0^\infty \phi(x + a)\, a^{\mu - 1} da = (-1)^\mu \overline{|\mu} \int^\mu \phi(x)\, dx^\mu,$$

* Airy's Tracts, p. 267. Traité de la Lumière, par C. H. D. Z. (Huygens), p. 17. A. Leide, 1690.

† Journal de l'Ecole Polytechnique, 21e Cahier, p. 8.

we obtain, if we write $\Sigma A z^{\mu-1}$ for $f(z)$, a for b^2;

$$\int_0^\infty z^{\mu-1}\,dz \sin\frac{2\pi}{\lambda}(vt-\sqrt{z+a}) = (-1)^\mu \lfloor\mu \int^\mu \phi(a)\,da^\mu$$

$$= (-1)^\mu \lfloor\mu \int^\mu \sin\frac{2\pi}{\lambda}(vt-\sqrt{a})\,da^\mu,$$

and consequently, from the equation above,

$$\Sigma A(-1)^\mu \pi \lfloor\mu \int^\mu \sin\frac{2\pi}{\lambda}(vt-\sqrt{a})\,da^\mu = \sin\frac{2\pi}{\lambda}(vt-\sqrt{a}).$$

This equation is satisfied (apparently) by making $\mu = 0$, $A = \frac{1}{\pi}$,

and consequently $f(z) = \frac{1}{\pi z}$, $f(r^2) = \frac{1}{\pi r^2}$.

The case is, however, of too doubtful a character to warrant us in adopting the conclusion. One thing alone I infer from it, that if any power of the distance (not of r) be assumed as the factor, it must be the inverse *square*. It would require that we should retrace our steps, and investigate the different formulæ corresponding to this hypothesis, before we could speak positively on the subject. I have only to add to this discussion on the probable coefficient of vibration, that an approximation has been made use of in the value of the distance between the disturbing and disturbed points, as it appears within the circular function. The approximation amounts in fact to supposing the wave *elliptical,* instead of circular. In the second problem I find that the square of this distance, being substituted within the circular function for the distance itself, leads to precisely the conclusions we have obtained. It is possible, therefore, that the omission of our factor, and the approximation made use of within the circular function, exactly counterbalance each other.

I cannot conclude without repeating my conviction of the importance of results such as those which Professor Forbes has just announced. It appears that the effect of scratching a piece of rock salt, &c. is to alter its power of transmitting heat in such a manner, that heat of a low temperature, or dark heat, is transmitted in greater proportions than before. If then the two kinds of heat correspond, the one to vibrations, or transmission due to vibrations; the other to transmission due to excess of elasticity, our analysis teaches us to expect that the quantity of the former kind stopped by the wires or scratches should be in exact proportion to the space covered by them, whilst we should hardly expect to find any considerable stoppage effected on the latter. Thus I am led to hope that the Theory which I proposed in the Transactions of the Cambridge Philosophical Society, Vol. vi., pp. 274, and seq. and subsequently developed in my little work on the subject, will be strengthened in some points, although I am far from expecting that it will be confirmed in all. Perhaps subsequent results may render it necessary to modify our hypotheses, but at present I do not know that experiment is very far in advance of theory. I cannot conclude without expressing my conviction that the masterly researches of Professor Forbes will have the effect of setting right several errors even in the Theory of Light, which have crept in from the difficulty of subjecting that branch of philosophy to strict measurement.

P. KELLAND.

EDINBURGH,
Jan. 23, 1840.

X. *On the Foundation of Algebra.* *By* AUGUSTUS DE MORGAN, F.R.A.S. F.C.P.S.; *of Trinity College; Professor of Mathematics in University College, London.*

[Read *Dec.* 9, 1839.]

THE extent to which explanation of the meaning of the symbolical results of Algebra has been carried within the last half century; the complete interpretation of all which formerly appeared incongruous; the separation, as it was called, of the symbols of operation and quantity, which amounts to the use of an algebra in which the symbols represent something more than simple magnitude;—will for some time to come suggest inquiry into the *logic* of this many-handled instrument of reasoning, which seems to be capable of presenting, under fixed laws of operation, all the results which arise from very distinct primary conceptions as to the things operated upon.

When several different hypotheses lead to results which admit of a common mode of expression, we are naturally led to look for something which the hypotheses have in common, and upon which the sameness of the method of expression depends. A comparison of the properties of the ellipse and hyperbola would bewilder the imagination, under any of the distinct definitions which might be given of the two curves; nor would the mind rest satisfied until it had discovered the reason of the similarity which exists between these properties.

Algebra now consists of two parts, the technical, and the logical. Technical algebra is the art of using symbols under regulations which, when this part of the subject is considered independently of the other, are prescribed as the definitions of the symbols. Logical algebra is the science which investigates the method of giving meaning to the primary symbols, and of interpreting all subsequent symbolic results. It is desirable that the word *definition* should not enter in two distinct senses, and I should propose to retain it as used in the *art* of algebra, applying the terms *explanation* and *interpretation* to denote the preparatory and terminal processes of the *science*. Thus a symbol is *defined* when such rules are laid down for its use as will enable us to accept or reject any proposed transformation of it, or by means of it. A simple symbol is *explained* when such a meaning is given to it as will enable us to accept or reject the application of its definition, as a consequence of that meaning: and a compound symbol is interpreted, when, having occurred as a result of explained elements, used under prescribed definitions, a necessary meaning can be given to it; the necessity arising from the tacit supposition that the compound symbol, considered as a new simple one, must still be subject to the prescribed definitions, when it subsequently comes in contact with other symbols. The last words may need the remark, that though we sometimes appear to interpret a symbol merely for the purpose of explaining a result, ye we know that such interpretation would be subsequently rejected, if the use of the symbol, under the prescribed definitions, were not found to be logically admissible.

A symbol is not the representation of an external object absolutely, but of a state of the mind in regard to that object; of a conception formed, for the formation of which the mind knows that it is or was indebted to the presence, bodily or ideal, of the object. Those who do not remember this, the real use of a symbol, are apt to dogmatize*, declaring one or another explanation of a symbol, that is, the signification by it of one or another impression produced on their own minds, to be real, true, natural, or necessary: it being neither one nor the other, except with reference to the particular mind in question. To take a very simple case, and one which bears upon our subject, let us imagine that we form successively a conception of the absence of all definite magnitude, followed by one of the existence of a certain magnitude, say a line of given length. The mind of one person may pass from the one to the other by imagining the given length to be instantaneously generated, no one portion of it coming into the thoughts before or after another; that of a second may make the transition by imagining a point to move from one extremity to the other: while that of a third may dwell rather on the relative position of the two extremities, and may think

* Of course, I use this word in its primitive sense, without any censure implied: the very sentence in which the word occurs is, and is meant to be, dogmatical.

more of B attained by motion from A, than of the quantity of length in AB. All three would use, perhaps, the same modes of expression: and I suspect* that there could be detected, among persons who think about first principles, a very considerable degree of variety in the points of view under which fundamental words suggest their objects; while as much exists, but could not as easily be found, among those who have studied the exact sciences, without paying particular attention to their foundations.

A symbol may thus denote either magnitude, operation, by which magnitude is attained, or the conception of one extreme arrived at, the other having been the previous object of contemplation. The earlier† algebraists most certainly dwelt on the first notion; $a + b$ is with them the result of an operation, in which the method of obtaining it is so completely forgotten, that the *result* $a + b$ is actually obtained by a distinct operation.

It seems to me that Sir William Hamilton, in his very original and methodical memoir on algebra as the science of pure time, has adopted a view of the third kind. I cannot see why the whole paper might not be as easily applied to succession of points in a line, as to succession of epochs in time. Succession, that is to say *continuous* succession, might be made the fundamental conception in both cases; and if such were the author's intention in the use of the word *time*, I should be very glad to maintain after him that *one* of the explanations which suffice to convert technical into logical algebra, has been fully established in his memoir. But, if any thing more *physical*‡ be intended by the distinguished author, and if some of his phrases are to be interpreted as of his asserting algebra to be *the* science of

* In a short biographical account (which I have before me, in a private communication) of the late Mlle Sophie Germain, whose papers on the theory of elastic surfaces are well known, it is asserted that she could never form the conception of space, except by the means of time: this was her own mode of expressing, to the writer of the notice, a state of mind by which he accounts for another fact, namely, that she had very little aptitude for pure geometry, and a great attachment to the theory of numbers.

† See my Calculus of Functions, sect. 245.

‡ This word is here improperly used; but I refer to the notion of those who would have made geometry a part of mixed mathematics: that is, if the algebra of Sir W. Hamilton would, in the opinion of those just alluded to, also have been a part of their mixed mathematics, and if Sir W. Hamilton should admit that they have as much reason, his terms being understood in his own sense, for their location of his algebra as for that of geometry, I should then say that the word used in the text is allowable.

pure time, I should then cite him as an instance of the *dogmatism* already alluded to: and the more readily, that by the association of the word with his labours, I may claim to have purified it, for the purposes of this paper, from the dyslogistic associations usually connected with it.

The modern algebraists usually dwell on the second notion, namely that of operation; and this I shall adopt in the present paper, not only as the most common mode of conception, but also as being equally capable of connexion with either of the other two. Imagine the process, whatever it may be, by which we pass from the contemplation of 0 to that of a; then if a represent a line, we can consider, as a result of our process, either the position of one extremity with respect to the other, or the quantity of length intercepted between the two.

I separate the following maxims from the rest as being equally applicable to the symbolical algebra which we have, and to any other which we might have. For it must never be forgotten that, though our present inquiry includes only the possible explanations of one given technical algebra, the subject may and probably must end in the investigation of others, or at least in the extension of the present one.

1. A simple symbol is the representative of one process, and of one only.

2. All processes, how many soever, may be looked at in their united effect as one process, and may be represented by one symbol.

3. Every process by which we can pass from one object of contemplation to another, involves a second by which we can reinstate the first object in its position: or every direct process has another which is its inverse. To complete the separation of these maxims from all others, I propose some considerations connected with the possible extensions of technical algebra.

The system of explanations which proceeds on the supposition that length affected by direction is the primary object of contemplation in algebra, is well known as to its history by Professor Peacock's Report to the British Association, and as to its present state by the Treatise on

Algebra of the same author*. But in this branch of logical algebra the lines must be all in one plane, or at least affected by only one modification of direction: the branch which shall apply to a line drawn in any direction from a point, or modified by two distinct directions, is yet to be found.

It is obvious that our power of making the preceding application of algebra is co-ordinate with that of assigning a symbol Ω, such that

$$a + b\Omega = a_1 + b_1\Omega \text{ gives } a = a_1 \text{ and } b = b_1.$$

An extension to geometry of three dimensions is not practicable until we can assign two symbols, Ω and ω, such that

$$a + b\Omega + c\omega = a_1 + b_1\Omega + c_1\omega \text{ gives } a = a_1,\ b = b_1 \text{ and } c = c_1:$$

and no *definite* symbol of ordinary algebra will fulfil this condition. Again, in passing from x to $-x$ by two operations, we make use in ordinary algebra of one particular solution of

$$\phi^2 x = -x, \text{ namely } \phi x = \sqrt{-1}\,.\,x.$$

An extension to three dimensions would require a solution of the equation $\phi^3 x = -x$, containing an arbitrary constant, and leading to a function of triple value, totally unknown at present.

A general solution of $\phi^2 x = ax$ can be expressed when any particular solution ϖx is known. For if $f\varpi f^{-1}x$ be the general solution, we have

$$\phi^2 x = f\varpi^2 f^{-1}x = f a f^{-1}x = ax, \text{ or } fax = afx:$$

so that it is only necessary that f and a should be convertible. Since then $(-1)^{\frac{1}{2}}x$ is a particular solution of $\phi^2 x = -x$, a general solution is $f\{\overline{-1}|^{\frac{1}{2}}f^{-1}x\}$ where $f(-x) = -fx$. But with our very limited knowledge of the laws of inversion, no solution which we can now express in finite terms will afford any help. Our means of expression must be augmented before we can hope to overcome this difficulty: or, as in most other cases

* Professor Peacock is the first, I believe, who distinctly set forth the difference between what I have called the technical and the logical branches of algebra. The second term, I am aware, is a very bad one, and I should be glad to see a better one proposed; but I prefer *technical* to *symbolical*, because the latter word does not distinguish the use of symbols from the explanation of symbols.

of the kind, our difficulties recur in a circle; the means which we have used to propound a possible method require the problem itself to be solved before they can be successfully used.

Let the object of contemplation be simple magnitude of any one kind, as in the arithmetic of concrete quantity. The process which must precede all others is what we call selecting one magnitude for consideration. Previously to this step, we have no object under our perceptions, and may write 0 as the representative of this preceding state, and as the recognition of its existence. This first magnitude we may call 1, and the operation of transition from one state to the other we may denote by $0 + 1$. The contemplation of simple existence, and of the possibility of expressing it by a spoken symbol, suggested the earliest definition of unity—ΜΟΝΑΣ ἐστι, καθ' ἣν ὃ ἕκαστον τῶν ὄντων ἓν λέγεται. If we represent our present state by $(0 + 1)$, we may consider that with respect to any other possible magnitude our position is what it was when we denoted it by 0. If we now denote it by $0'$, we may, as before, make the transition from $0'$ to $0' + 1$, which implies that we have further taken into consideration a new magnitude of the same amount.

This result, $(0+1)+1$, we may, if we please to consider it as attained by one operation, signify by $0+2$: and so on. Using the symbol $-$ to denote the process by which we retrace our steps, we have all that is necessary to express addition and subtraction. The principle which I wish here to enforce is, that *addition is connected with the symbol* 0 *in a manner which requires us to imagine that we start from one magnitude, as it were from a new* 0, *and renew* the process by which we passed from the first* 0 *to that magnitude.*

Let us now suppose that modified magnitude is under contemplation, and let the simple symbol a denote a line measured in a given direction from the zero point 0. In this zero of space, which admits of an infinite number of positions, we seize more clearly than before that notion which, as to simple magnitude, is not easily admitted as necessary, and may seem rather fanciful: namely, that every magnitude attained may, as

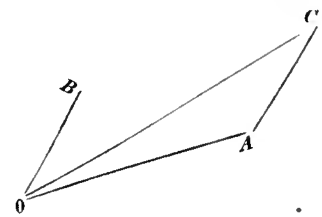
* Any one who doubts the justness of this fundamental position should add six to four on his fingers, having previously refreshed his notions of six and four by the same process.

to future addition, be considered as a new zero. We are now to assume that,

1. Parallelism and sameness of direction are meant to be identical terms; that is to say, the two directions conceivable on any one of two parallels are severally the same as the two directions on the other.

2. Every simple symbol represents a line given in length and direction: thus $a = b$ means that the lines a and b, equal in length, have also the same direction. And the process implied in $0 + a$ is the transference of a point from the position 0 to a given length in a given direction.

We can now find the necessary meaning of $(0 + a) + b$; *necessary*, on the supposition that the technical algebra is to become logical on the explanation of the symbols before us. Let $0A$ and $0B$ represent the lines symbolized by a and b: if then we take A, at which we arrive by the process $0 + a$, as a new zero, and proceed with it in the same manner as in performing $0 + b$ on the old zero, we draw AC parallel and equal to $0B$, whence $0C$ being symbolized by c, we have with reference to the first zero,



$$0 + c = (0 + a) + b = (0 + b) + a.$$

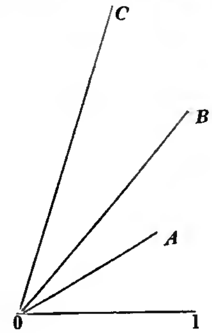
I need not further dwell on the connection of addition and subtraction in arithmetic with the processes called by the same names in this explanation. I shall only here suggest that perhaps the words *direct zero process* and *inverse zero process* might occasionally be found useful*.

The usual method of defining the process of addition by reference to the diagonal of a parallelogram is convenient, but destructive of all true analogy. The fundamental theorem of statics suffers from the same method of statement.

I now proceed to the process of multiplication, which will readily be seen to be connected with *unity* in precisely the same manner as is addition with zero. If b be formed from unity by the train of processes $0 + 1 + 1 + 1$, we consider a as a new unit, and let the symbol ba represent

* In my *Calculus of Functions* (sect. 12, 13, 17) will be found some analogies connecting simple addition with zero, and multiplication with unity.

the same operation on this new unit, or $0 + a + a + a$. Similarly, if by the line 1 we mean a line having the length and direction 1, and $0A$ and $0B$ by a and b, and if we take $0A$ as a new unit, and perform on it the operations by which we pass from 01 to $0B$, that is, take an angle $A0C$ equal to $10B$, and let $0C$ be in length the result of the arithmetical operation on $0A$ and $0B$,—then $0C$ must be represented by ab. The processes of multiplication and division might be called the direct and inverse *unit processes*.



There is now nothing particular to be said about the four operations, or the simple powers, with positive or negative, whole or fractional, real exponents, or any combinations of them. The interpretation of $a + b\sqrt{-1}$ follows in the usual manner.

In illustration of the propriety of considering symbols as functions of zero or unity for purposes of addition or multiplication, it may be advanced that unless we do so, we change the meaning of the terms direct and inverse as we proceed from the lower to the higher parts of the science. Unquestionably, if ever we have a right to assume a clear conception of this distinction, it is in the comparison of addition with subtraction, and of multiplication with division; but for all that, $a + x$ and $a - x$ are not inverse functions, considered with respect to x, though they are so with respect to a. And similarly of ax and $a \div x$. When we come to the symbol x^n, then, and then only, do we begin to describe inversion correctly: for we usually consider this as a function of x and not of n, when we assert $x^{\frac{1}{n}}$ to be the inverse. But if we considered this as a function of n, the inverse would be $\log n : \log x$.

The separation, as it is called, of the symbols of operation and quantity, is a method of explaining technical algebra as simple in its character as the preceding. Let the fundamental object of conception be $\phi(x - nh)$, n being infinite, which stands in the place hitherto occupied by 0. Let* $\Sigma\phi(x + ah)$ represent the train of operations by which we pass from $\phi(x - \infty h)$ to $\phi(x + \overline{a - 1}h)$, or

* In the common method of treating this subject, the inverse symbol is made to precede the direct one. Several adaptations of notation are necessary before we can exactly represent the common methods.

$$\phi(x - \infty h) + \ldots\ldots + \phi(x - h) + \phi x + \phi(x + h) + \ldots\ldots + \phi(x + \overline{a - 1}h).$$

The inverse operation, or rather the operation by which $\phi(x + ah)$ is obtained from $\Sigma\phi(x + ah)$, is either $\Sigma\{\phi(x + \overline{a + 1}h) - \phi(x + ah)\}$, or $\Sigma\phi(x + \overline{a + 1}\,h) - \Sigma\phi(x + ah)$, and may be symbolized either by $\Delta\Sigma\phi(x + ah)$ or $\Sigma\Delta\,\phi(x + ah)$.

The proper way, however, of considering this class of extensions may not be as a simple explanation of technical algebra, (though it might be regarded in that point of view,) but as an extension of technical algebra itself, in which new explanations of the direct and inverse unit process are used co-ordinately with the one already established. If we agree to signify by ∇^0, ∇^1, ∇^2, &c. a new progression of operations, in which the zero and its processes remain subject to the usual definitions, nothing prevents us from supposing that the prescribed definitions of the unit process may remain true if ∇^0 be made the unit, ∇^2 being derived from ∇^1 by the same train of operations as ∇^1 from ∇^0, and so on. Neither is it impossible that the same laws of convertibility and distribution may exist between compound operations, in which different units are employed, as are laid down in the prescribed definitions relatively to the different unit processes suggested by simple magnitudes.

Let $\nabla^0 = \phi x$, and

$$\nabla^1 = a_0\phi x + a_1\phi(x + h) + a_2\phi(x + 2h) + \ldots\ldots$$

where a_0, a_1, &c. may be functions of h, but not of x. Technical algebra may be carried to its full length under these explanations, and many developements may be and have been simplified by their means. It is not my intention here to write a treatise on this subject: my object is, to point out *that the logic of each and all of these explanations is the same; no mode of arriving at any one explanation differing from that of any other in the fundamental, and what we may call the arithmetical, part of the subject.* It is certain that the discovery of inverse operations is not yet complete: this must be reserved until such time as the branches, which adopt length modified by direction as the explanation of simple symbols, are properly connected with that technical algebra, in which various unit processes are used co-ordinately with the same zero process.

It may perhaps be worthy of note that the series

$$a_0 + a_1 x + a_2 x^2 + \ldots\ldots$$

may be considered as $\nabla \epsilon^{\upsilon}$ when $\upsilon = 0$ in the equation

$$\nabla \epsilon^{\upsilon} = a_0 \epsilon^{\upsilon} \times a_1 \epsilon^{\upsilon + \log x} + a_2 \epsilon^{\upsilon + 2\log x} + \ldots\ldots$$

I now return to the purely algebraical question. It is in our power to avoid all ambiguity in results, by simply prescribing that every symbol shall express not merely the length and direction of a line, but also, the quantity of revolution by which a line, setting out from the unit line, is supposed to attain that direction. When this is done, I shall use a double sign of equality to denote it. Thus, if we denote by (a, θ) a line of a length a, which has made the revolution θ, it is allowable to write

$$(a, \theta) = (a, \theta + 2\pi) = (a, \theta + 4\pi), \ldots\ldots$$

but not

$$(a, \theta) = = (a, \theta + 2\pi) = = (a, \theta + 4\pi) \ldots\ldots$$

As long as we neglect this additional prescription, great care will be requisite to prevent our falling into error. While exponents transform lengths into lengths, and directions into directions, no great caution is requisite : but when, as we shall presently see, an exponential process causes the exponent of a length to affect that of direction, or *vice versâ*, the following fallacy of a continental analyst, mentioned by Professor Peacock in his Report, is frequently likely to occur. Stripped of unnecessary details, it is as follows :

$$\epsilon^{2\pi n\sqrt{-1}} = 1, \ (\epsilon^{2\pi n\sqrt{-1}})^{2\pi n\sqrt{-1}} = 1^{2\pi n\sqrt{-1}} = 1,$$

$$\text{or } \epsilon^{-4\pi^2 n^2} = 1, \text{ an absurd result.}$$

The answer is very simple: if no extension of explanations be contemplated, $1^{2\pi n\sqrt{-1}}$ is not necessarily $= 1$, since it may have an infinite number of values. If the extensions be made, and if $=$ merely denote sameness of direction, the same thing is true; for $1^{2\pi n\sqrt{-1}}$ or $(\epsilon^{2\pi n\sqrt{-1}})^{2\pi n\sqrt{-1}}$ has an infinite number of values, of which one only $(\epsilon^0)^{2\pi n\sqrt{-1}}$ is $= 1$: and the same fallacy might be thus propounded;

$$\sqrt{x^2} = + x, \ \sqrt{x^2} = - x, \text{ therefore } x = - x.$$

But if = imply sameness of revolution, it is not true that $\epsilon^{2\pi n\sqrt{-1}} = 1$, except in length.

The interpretation of $A^{\sqrt{-1}}$ might be easily attained from prescribed definitions, and from their necessary result

$$\epsilon^{\theta\sqrt{-1}} = \cos\,\theta + \sin\theta\sqrt{-1}\,;$$

nor would this step be logically objectionable. It would, however, be more satisfactory if something like an *à priori* interpretation, or simple explanation, could be given. I do not consider the following as complete, but it is, as far as it goes, of a new character.

Conformably to definitions, we must have

$$\{(\log a,\,\theta)^{\sqrt{-1}}\}^{\sqrt{-1}} = \{\log a,\,\theta\}^{-1} = (-\log a,\,-\theta),$$

where by $(\log a, \theta)$ is meant a line of the length a, and amount of revolution θ. Now we cannot suppose that the first operation changes the sign of $\log a$ only, and the second that of θ only: for this would be to make the operation $(\quad)^{\sqrt{-1}}$ mean different things in different places. We must propose some operation of permanent form, which being twice performed will make the alteration required.

From the definitions, it follows that

$$(\log a,\,0) \times (0,\,\theta) = (\log a,\,\theta),$$

whence $(\log a, \theta)$ must be the product of two functions, one of a and the other of θ, the first of which is known, being $\epsilon^{\log a}$ or a, and the second of which must be of the form E^{θ}, since by definition

$$(0,\,\theta) \times (0,\,\theta') = (0,\,\theta + \theta').$$

Hence aE^{θ}, or $a(0,1)^{\theta}$, is the representative of a line a, inclined at an angle θ. If then we make $\cos\theta$ and $\sin\theta$ mean nothing more than the projecting factors of a length inclined at the angle θ upon the axis of the unit line and its perpendicular, we have

$$(\cos 1 + \sqrt{-1}\,\sin 1)^{\theta} = \cos\theta + \sqrt{-1}\,\sin\theta.$$

The definition does not differ from that of $\cos\theta$ and $\sin\theta$ in geometry, and this equation is an *à priori* property of these functions, deducible immediately from the definition, in any system which gives meaning to $\sqrt{-1}$ from its commencement.

The hardest and most delicate part of this investigation is the connexion of $\epsilon^{\theta\sqrt{-1}}$ with a unit inclined at an angle θ; or generally to show that the operation $(\quad)^{\sqrt{-1}}$ changes the exponent of length into one of direction, and *vice versâ*, without the necessity of inferring this from interpretation. If we assume beforehand that $\epsilon^{\sqrt{-1}}$ is *real*, under the extended definitions, it would be difficult to imagine what other office $(\quad)^{\sqrt{-1}}$ could perform; but such an assumption would not be a proper one, since all the associations of preceding algebra would lead us to suppose that each extension removes only one class of inexplicables, and leaves, or perhaps introduces, others. I cannot complete this part of the subject satisfactorily, but the following considerations will show that the most simple mode of attaining, upon an explanation, the technical end of the operation $(\quad)^{\sqrt{-1}}$ is precisely that which answers to the above.

Required an operation which repeated n times upon a function of n quantities shall end by changing the sign of all. Take four quantities, a, b, c, and d. Successive changes of sign made upon one after the other will be really different successive operations; but if we change the sign of a given one, say the first, and at the same time make a set of periodic interchanges, writing b for a, c for b, d for c, and a for d, we shall have an operation which repeated four times will produce the desired effect. Thus we have successively,

$$\phi(b, c, d, -a),\quad \phi(c, d, -a, -b),\quad \phi(d, -a -b, -c),\quad \phi(-a, -b, -c, -d).$$

Thus we see in the succession $(\log a, \theta)$, $(-\theta, \log a)$, $(-\log a, -\theta)$ a method of passing from A to A^{-1} at two similar steps, which does not involve the use of $\sqrt{-1}$. We see the same in $(\log a, \theta)$, $(\theta, -\log a)$, and $(-\log a, -\theta)$. If then we assume, as a suggestion,

$$(\log a, \theta)^{\sqrt{-1}} = (-\theta, \log a),\quad (\log a, \theta)^{-\sqrt{-1}} = (\theta, -\log a),$$

we find, making $A = (\log a, \theta)$, the following equations;

$$(A^{\sqrt{-1}})^{\sqrt{-1}} = A^{-1},\quad (A^{-\sqrt{-1}})^{-\sqrt{-1}} = A^{-1},\quad (A^{\sqrt{-1}})^{-\sqrt{-1}} = A,$$

$$\left(A^{\frac{1}{\sqrt{-1}}}\right)^{\frac{1}{\sqrt{-1}}} = A^{-1},\quad \left(A^{-\frac{1}{\sqrt{-1}}}\right)^{-\frac{1}{\sqrt{-1}}} = A^{-1},\quad \left(A^{\frac{1}{\sqrt{-1}}}\right)^{-\frac{1}{\sqrt{-1}}} = A,$$

in perfect fulfilment of all the fundamental conditions which prescribed definitions impose. The assumption gives

$$(aE^{\theta})^{\sqrt{-1}} = E^{\theta\sqrt{-1}} \cdot \epsilon^{\log a \cdot \sqrt{-1}},$$

where $E^{\theta\sqrt{-1}}$ must be a symbol of length, and $\epsilon^{\log a \sqrt{-1}}$ of a unit inclined at the angle $\log a$. Consequently $\epsilon^{\theta\sqrt{-1}}$ must signify a unit inclined at an angle θ.

It might be asked whether there is anything in the preceding process which restricts us to the use of the base ϵ rather than any other, I answer, nothing whatever: but at the same time there is nothing which binds us to the use of any particular method of measuring angles. It may be deduced from the preceding that the base ϵ must be used co-ordinately with that mode of measurement which I call *theoretical**. This connexion depends entirely upon the purely numerical process by which the equation $\epsilon^{2\pi\sqrt{-1}} = 1$ is proved to be satisfied when ϵ and π have their usual meanings. If for any reason we prefer the base a, the measure of two right angles must be $\pi \times \{\log \epsilon \text{ to the base } a\}$.

I think it cannot be disputed that interpretation should be avoided where explanation can be given. If where the latter cannot be obtained suggestion upon such analogies as present themselves were to take its place, the former would be also replaced by verification. In the present instance, the attainment of

$$\epsilon^{\theta\sqrt{-1}} = \cos\theta + \sqrt{-1}\sin\theta \text{ from } E^{\theta} = \cos\theta + \sqrt{-1}\sin\theta$$

is the verification.

I now come to the theory of logarithms. It is a circumstance which I hold to be not a little remarkable, that the ancient form of algebra was only saved from being convicted of incapacity to produce its own legitimate results, but very little time before such an escape would have been rendered impossible by its receiving the necessary accession from the more extended form. Mr Graves has admitted that his view of the new logarithms should rather have been that of an extension imperatively

* In those works on Trigonometry which use the arc and angle indiscriminately, this mode of measurement is said to be *in parts of the radius*. A term is much wanted which shall not imply this confusion between arcs and angles; and I propose that the angle which subtends an arc equal to the radius shall be called the *theoretical unit*.

required than of a correction to already existing formulæ: and in this view I perfectly agree. If we define log x, or rather λx, (reserving log x for the numerical logarithm of the length) to be any legitimate solution of $\epsilon^{\lambda x} = x$, it is plain that the logarithm of n inclined at an angle ν, (or of N) to the base b inclined at an angle β, (or B) is to be derived (avoiding ambiguity) from

$$(b\epsilon^{\beta\sqrt{-1}})^x = = n\epsilon^{\nu\sqrt{-1}},$$

$$\text{or } \lambda_B N = = \frac{\log n + \nu\sqrt{-1}}{\log b + \beta\sqrt{-1}}.$$

This result is real when $\dfrac{\log n}{\log b} = \dfrac{\nu}{\beta}$; nor is it more surprising that an impossible quantity (hitherto so called) should have a possible logarithm, than that exponential operations not containing $\sqrt{-1}$, or not interchanging exponents of length and direction, should in certain cases enable us to pass from one line to another. I need not enter into details of the properties of the preceding equation. If we admit all symbols to be algebraical (in the old sense) which denote lines drawn in the unit line or its continuation, whatever may be the number of complete revolutions after which they rest there, we must then admit that the logarithm of a unit which is in its position for the $(m+1)^{\text{th}}$ time, with respect to ϵ which is in its position for the $(n+1)^{\text{th}}$ time is

$$\lambda_{(0,\,2n\pi)}(0,\,2m\pi) = \frac{2m\pi\sqrt{-1}}{1+2n\pi\sqrt{-1}}$$

the form proposed by Mr Graves.

In a work of M. Cauchy, and perhaps in other writings which I am not acquainted with, mention is made of a singular point in curves which he calls *point d'arrêt*, at which the branch suddenly stops. Such a point has long been admitted in the spiral of Archimedes and other curves, owing to the neglect of making those extensions with regard to the sign of the radius vector which were necessary to complete the connexion* of polar and rectangular co-ordinates; and from the assumption of the impossibility

* On this subject I may be allowed to refer to page 341 of my Treatise on the Differential Calculus.

of which (I speak from memory) D'Alembert drew those instances in which he contended that the negative quantity is not *always* the contrary of the positive quantity. Disregarding such *points d'arrêt*, there is another sort which frequently occurs (but only in exponential or logarithmic curves), in which the abruptness of the termination is better marked. Thus in $y = (1 - x) \log (1 - x)$, there is, in our present system, an absolute cessation of the curve when $x = 1$ and $y = 0$. Here, when the requisite extensions of the logarithmic theory are made, it will be seen that there is not an absolute abrupt termination, but the commencement of what French writers have called a *branche pointillée*, a part of a curve, which I do not remember to have seen mentioned in any English work, except Professor Peacock's Report.

A. DE MORGAN.

University College, London,
October 16, 1839.

XI. *On the Effect of the Non-Residence of Landlords, &c. on the Wealth of a Community.* *By* J. TOZER, ESQ. M.A. *Caius College.*

[Read *March* 16, 1840.]

THE investigations that have been made by political economists of the effects produced on the wealth of a community by the non-residence of its proprietors, have frequently been asserted to furnish results which are not confirmed by observation. The following is believed to be a more careful investigation of the problem than any that has yet been made, and one that accounts for the apparent discrepancy.

While the proprietor resides, his income, subject to such deductions as are made by direct taxation, will be expended either in purchasing commodities or in paying for services. Those whose services he retains will expend what they receive in the same manner, and therefore the whole income of the proprietor will be expended, either directly or indirectly, in the purchase of commodities. The necessary and sufficient division of these will be into two classes, those which have been produced by the labour and capital of the countrymen of the proprietor, and those which have been produced by the labour and capital of foreigners, and which have been placed within his reach by the employment of capital and labour which may have been either native or foreign.

Of taxes, we need only consider those which the proprietor is constrained to pay while he is resident, and whose payment he evades by non-residence. We may also, without affecting the result, assume these to be paid when the income is realized, and not before.

We have then, while the proprietor resides, a portion of capital employed in raising the produce of his estates; a second portion in raising such other native commodities as are consumed by himself or those whose services he retains, and in distributing such foreign productions as are so consumed; and a third portion, which is commercial capital, belonging either to resident natives or to foreigners, and which is employed in purchasing and importing those foreign productions.

Each of these classes is susceptible of the usual further divisions into fixed and floating.

Call these portions of capital $C_1\,C_1'$, $C_2\,C_2'$, $C_3\,C_3'$ respectively; $C_1'\,C_2'\,C_3'$ being fixed, and $C_1\,C_2\,C_3$ floating.

Then if q be the amount of a unit of capital with its profit, C_1, C_2, C_3, must at the end of the year, or other period of return, yield $q\,C_1, q\,C_2, q\,C_3$, respectively, and C_1', C_2', C_3' must yield annuities which pay the profits $(q-1)C_1'$, $(q-1)C_2'$, $(q-1)C_3'$, and replace such portions of those capitals as have been destroyed. Call these annuities A_1, A_2, A_3, respectively.

$$\text{Then if } q\,.\,C_1 + A_1 = q(c_1 + c_2 + \dots c_n) = q\Sigma c,$$

where $c_1, c_2 \dots$ are portions of capital which respectively yield the returns $r_1, r_2 \dots r_n$ for each unit, the fixed capital being expressed in terms of its value as floating capital, and $r_1, r_2 \dots r_n$ being a decreasing series, the proprietor's income will be

$$\Sigma_n^1 c(r-q), \text{ which suppose } = R;$$

Σ_n^1 indicating the sum of the terms $c_1(r_1-q)\;c_2(r_2-q)\dots c_n(r_n-q)$.

Of this, a part Rt suppose will be paid in those direct taxes the payment of which will be evaded by non-residence.

A part $q\,C_2 + A_2$ will be received by the owners of $C_2 + C_2'$.

The remainder $q\,.\,C_3 + A_3$ by the owners of $C_3 + C_3'$. Let this remainder $=\Sigma\rho$; ρ_1, ρ_2, &c. being portions of it which are exchanged for portions of foreign produce of different kinds.

Suppose now H to be the country in which the proprietor's lands are situated, K that in which the equivalent for ρ_1 is produced; and let σ_1 be the price of that equivalent in K.

Then if there be direct intercourse between H and K, and if K import the produce of the proprietor's estates, this price in H must have been raised to $\sigma(1+\epsilon+\eta)$ where $\epsilon\sigma$ is the expence of making the transfer, and $\eta\sigma$ the import duty in H, it being supposed that there are no export duties. If K do not import the produce of the proprietor's estates, she will either import some other articles produced in H, or the produce of some other country which does import from H; and there will be a series of exchanges effected by equivalents that are produced by capital other than that we are considering, the last operation of importing to H yielding a tax $\tau\sigma$ to its revenue.

Hence while the proprietor remains at home the produce of his estate is thus distributed:—

To native capitalists and labourers, $\Sigma_n^1 c\{r(1-t)+qt\} - \Sigma\rho$, of which $C_1 + C_2 + C'_1 + C'_2 + A_1 + A_2 - (C'_1 + C'_2).q$ is employed in replacing capital.

To revenues of foreign states, $\Sigma\sigma\Sigma\eta$.

To revenue of H, $t\Sigma_n^1 c(r-q) + \tau\Sigma\sigma$.

To commercial capitalists and labourers, $\Sigma\sigma\Sigma\epsilon$; the remainder of $\Sigma\rho$ replacing the commercial capital with which the foreign purchases were made.

When the proprietor becomes non-resident the capital $C_2 + C'_2$ will be disengaged, because his absence destroys the demand on which its employment depended; but a new demand for such commodities as can be exported with advantage will be created by the absence, because the rent of the proprietor must now be exported. It is therefore in raising such commodities that the disengaged capital will be employed. If then the exports of H be made in manufactured goods, there will not in general be any variation in the rate of profit, because the employment of additional capital in manufacturing does not diminish profits, if the demand for manufactures be proportionably increased.

The income to support the inhabitants of H is therefore precisely the same before as after the removal; but there is this difference, the equivalent for the produce of $C_2 + C'_2$ was then possessed by the proprietor, it has now to be created by the labour of those who produce the commodities, by the exportation of which his rent is paid.

Again, let the exports of H consist of raw produce, then to the series $c_1\, c_2 \ldots c_n$ there will be in general added the terms $c_{n+1}\, c_{n+2} \ldots c_{n+m}$, producing respectively the returns $r_{n+1} \ldots r_{n+m}$, an effect which will not be confined to the estate of the proprietor who has become an absentee. The rate of profit will be lowered from $q-1$ to $q'-1$, where q' is determined by the least return which is now made by an unit of capital; the whole rental of H will be increased, and the income of the proprietor will be raised from

$$R = \Sigma_n^1 c(r-q), \text{ to } R_1 = \Sigma_{n+m}^1 c(r-q')$$

$$\text{whence } R' - R = \Sigma_{n+m}^{n+1} c(r-q') + (q-q')\Sigma_n^1 c,$$

which will be the gain of the proprietor. The treasury of H will lose $t\Sigma_n^1 c(r-q) + \tau\Sigma\sigma$. Foreign treasuries will gain $R'\Sigma\eta' - \Sigma\rho\Sigma\eta$, $R'\eta_1'$, &c., being the import duties successively paid on R' and its equivalents.

If the same amount of commercial capital be required to export R' as was required to export $\Sigma\rho$, and import its equivalent, $C_3 + C'_3$ will be unaffected; but if a different amount, and if the owners of this capital be residents of H, then a portion of $C_3 + C'_3$ may be diverted to the same employments as $C_2 + C'_2$ have been compelled to seek; or a portion of $C_2 + C'_2$ may be employed as commercial capital. The effects of non-residence will thus be increased or diminished in degree, but will continue to be the same in kind.

The distinction between a country which exports manufactures and one that exports raw produce is not a necessary one, though it may generally exist. The accurate enunciation of the result appears to be, that beyond the loss to the revenue the absenteeism of proprietors can only impair the resources of a community when it forces capital from a more to a less profitable employment.

As regards the effect on F, the country to which the proprietor has removed, the nature of the products in which his rent will be imported is independent of the nature of those he consumes, as well as of those in which it was produced. The presence of an individual who is without capital, but who is entitled to an income, will therefore create a demand for the employment of capital in producing the commodities he requires.

This capital will be drawn from investments where its employment produces the least return. There will be a gain in direct taxes paid by the immigrant, and beyond this there will be a gain, if supplying his wants afford a more profitable employment for capital than that from which it is withdrawn. This will not necessarily be the case, since the importation of the income may have rendered some species of home production unprofitable, and the demand for capital being thus accompanied by its disengagement, the rate of profit may be unaffected. In the case in which F imports raw produce, there will be an abstraction of capital from its production at home.

If then $c_1 c_2 \dots c_\nu$ be portions of capital, the units of which yielded before the withdrawal $r_1 r_2 \dots r_\nu$ respectively of produce, $r_1 r_2 \dots r_\nu$ being a decreasing series, and if $c_{\mu+1} \dots c_\nu$ be withdrawn, the rate of profit being changed from q to q_1, the whole rental of F will be reduced from

$$\Sigma_\nu^1 c(r-q), \text{ to } \Sigma_\mu^1 c(r-q_1),$$

and therefore diminished by

$$\Sigma_\nu^{\mu+1} c(r-q) + (q'-q)\Sigma_\mu^1 c.$$

If the imports of F be made in manufactures, there will not in general be any increase of wealth consequent on the residence in F of the proprietor whose home is H, beyond the direct taxes he pays; but there will be a substitution of a portion of income obtained without labour for an equal portion obtained by labour.

The reason why these results contradict general opinion on the subject is this: we are compelled, as preliminary to our investigations, to limit by definition the meaning of the terms we employ. The proprietor, in the vocabulary of the political economist, is simply the individual entitled to a certain income when it can be realized, and constantly either anticipating that income, or devoting it when received to the purchase of products whose creation rendered the employment of the capital of others necessary; his absence therefore does not involve any removal of capital, and consequently does not diminish the means of supporting human life. It has also in this investigation been supposed that he and his dependents expend the whole income to which he is entitled, without leaving any trace of that expenditure in the shape of capital accumulated, with a view either to durable or prospective improvement, or to profit. This supposition must have been tacitly made whenever the wealth of a country has been pronounced to be unaffected by the non-residence of its landlords.

Let us now suppose a portion of the proprietor's income to have been, by himself or by his dependents, either accumulated or itself employed as capital in producing the commodities for which it is exchanged. Call this portion a, and let a part of it $(1-l)a$ be expended without accumulation, and a part la be employed as capital with a view to profit. If then the rate of profit continued constant, and the whole of the proceeds of la were employed as capital, we should have had in the $x+1^{\text{th}}$ year to expend on labour, instead of a,

$$a\left\{l\frac{q^x-1}{q-1}+(1-l)\right\};$$

and the fund for employing labour will therefore have been at the commencement of this year diminished by $qla.\dfrac{q^{x-1}-1}{q-1}$.

Or, if we suppose the rate of accumulation in any one year to be changed to a v^{th} part of what it was in the preceding, from an alteration in the rate of profit, or from a different proportional part of that which is produced being accumulated; the fund for employing labour will be at the end of the first, second,.... years, instead of

$a, a\{1+lq\}, a\{1+l(q+q^2)\}$, &c. $a, a(1+lqv), a\{1+l(qv^2+q^2v^{1+2})\}$, &c.;

and the loss to this fund during the $x+1^{\text{th}}$ year would be

$$a\{1+l\Sigma(qv^{\frac{x+y-1}{2}})^{x-y}\},\ y \text{ to be taken from } x-1 \text{ to } 1.$$

This expression may be made to include the loss of the proprietor's services as a skilled labourer.

But further, a consequence of the absence of the proprietor may be the removal of a portion of the capital which that absence has forced into new investments, and the destruction of another portion. Of the capital C_2+C_2', then, let the part mC_2+nC_2' be removed without a change of residence of the owners, and the part $m'C_2+n'C_2'$ be either destroyed or removed in such a way, that its profit is no longer expended in H; and let the fractional parts λ, λ' respectively of these have been employed in producing capital, the remainder having been expended without accumulation.

Then the resultant loss to the community from the latter part will be

$$(1 - \lambda)\ \{m C_2 + n\,[A_2 - (q - 1)\,C_2']\} + (1 - \lambda')\,(m' C_2 q + n'.A_2);$$

and from the former part, at beginning of 1st year,

$$\lambda m C_2 + \lambda' m' C_2;$$

at beginning of 2nd year,

$$\lambda m C_2 . q + \lambda' m' C_2 . q + \lambda n\,\{A_2 - (q - 1)\,C_2'\} + \lambda' n_1 A_2;$$

at beginning of 3rd year,

$$\lambda m C_2 q^2 v + \lambda' m' C_2 q^2 v + \lambda n\,\{A_2 q - (qv - 1)\,C_2' q\} + \lambda' n_1 A_2 q v;$$

....

at beginning of $x + 1^{\text{th}}$ year,

$$\{(\lambda m + \lambda' m')\,C_2 q v^{x-1} + (\lambda n + \lambda' n' v^{x-1})\,A_2 - \lambda n\,(q v^{x-1} - 1)\,C_2'\}\,(q v^{\frac{x-2}{2}})^{x-1}.$$

Hence, during the $x+1^{\text{th}}$ year, the whole possible loss of income will be:—

From expenditure of Proprietor, $a\{1 + l\,\Sigma_{y=x-1}^{y=1}(q\,v^{\frac{x+y-1}{2}})^{x-y}\}$.

From Capital removed or destroyed whose profits would have been expended without accumulation,

$$(m + m'q)\,C_2 - n\,(q - 1)\,C_2' + (n + n')\,A_2 + \lambda\,\{n\,(q - 1)\,C_2' - m\,C_2 - n A_2\} - \lambda'\,(m' C_2 q + n' A_2).$$

From Capital removed or destroyed whose profits would have been accumulated,

$$\{\lambda\,[(m\,C_2 - n\,C_2')\,q v^{x-1} + n\,(C_2' + A_2)] + \lambda'\,(m' C_2 q + n_1 A_2) v^{x-1}\}\,(q v^{\frac{x-2}{2}})^{x-1}.$$

The greatest possible loss in this year, when v, l, m', n', λ', each $= 1$, and $\therefore m, n$ each $= 0$, will be

$$a\,.\frac{q^x - 1}{q - 1} + q^{x-1}\,(C_2 q + A_2);$$

and the least when l, λ, λ', each $= 0$,

$$a + (m + m_1 q)\,C_2 - n\,(q - 1)\,C_2' + (n + n')\,A_2.$$

As far as the destruction of Capital is concerned the investigation, of course, applies to the case of proprietors becoming absentees, and not of their continuing so.

The general effects of absenteeism may be thus enunciated. There will in all cases be a loss to the home-revenue in those direct taxes whose payment can be evaded by absence. There will, whenever there are duties on importation and not on exportation, be a further loss of the import-duty paid on the foreign productions which the proprietor consumed when at home.

Beyond this there will be a diminution of the aggregate income of the community whenever the capital that is disengaged by the absenteeism is forced into less profitable employments than those it previously occupied, and in no other case. When therefore the country from which the proprietor absents himself exports raw produce, there will generally, though not necessarily, be a loss beyond the loss to the revenue, and this loss will be accompanied by a general increase of rental. When it exports manufactures, there will not in general be any loss beyond that to the revenue.

There will however in all cases be this further and very important effect: though the income which the proprietor removes may be replaced, it must be replaced by labour, and there will therefore be substituted for the leisure class, which a part of that income maintained, a class who must by their own exertions produce the incomes on which they subsist; and there is nothing in the conditions of the problem to limit the extent to which the subdivision of income may, among the members of this class, be carried, or to fix the minimum that may be enjoyed by each.

It is necessary to the truth of these results, that the withdrawal of the proprietor should cause no removal of capital, that any part of the proprietor's income which was not expended should still be saved at home, and that no part should have been consumed without calling for the employment of capital.

In applying the result to any particular country, the first step is to decide how far, in the case of its absentees, these conditions are fulfilled; if they be not fulfilled, or if the individuals who remove had in any degree the qualities of productive labourers, the wealth of the community must be impaired by their absence; and the injury is capable of increasing with time to an indefinite extent.

J. TOZER.

CAIUS COLLEGE, CAMBRIDGE,
March 16, 1840.

XII. *Demonstration that all Matter is heavy. By the Rev.* WILLIAM WHEWELL, B.D. *Fellow of Trinity College and Professor of Moral Philosophy.*

[Read *February* 22, 1841.]

THE discussion of the nature of the grounds and proofs of the most general propositions which the physical sciences include, belongs rather to Metaphysics than to that course of experimental and mathematical investigation by which the sciences are formed. But such discussions seem by no means unfitted to occupy the attention of the cultivators of physical science. The ideal, as well as the experimental side of our knowledge must be carefully studied and scrutinized, in order that its true import may be seen; and this province of human speculation has been perhaps of late unjustly depreciated and neglected by men of science. Yet it can be prosecuted in the most advantageous manner by them only: for no one can speculate securely and rightly respecting the nature and proofs of the truths of science without a steady possession of some large and solid portions of such truths. A man must be a mathematician, a mechanical philosopher, a natural historian, in order that he may philosophize well concerning mathematics, and mechanics, and natural history; and the mere metaphysician who without such preparation and fitness sets himself to determine the grounds of mathematical or mechanical truths, or the principles of classification, will be liable to be led into error at every step. He must speculate by means of general terms, which he will not be able to use as instruments of discovering and conveying philosophical truth, because he cannot, in his own mind, habitually and familiarly, embody their import in special examples.

Acting upon such views, I have already laid before the Philosophical Society of Cambridge essays on such subjects as I here refer to; especially a memoir "On the Nature of the Truth of the Laws of Motion," which was printed by the Society in its Transactions. This memoir appears to have excited in other places, notice of such a kind as to shew that the minds of many speculative persons are ready for and inclined towards the discussion of such questions. I am therefore the more willing to bring under consideration another subject of a kind closely related to the one just mentioned.

The general questions which all such discussions suggest, are (in the existing phase of English philosophy) whether certain proposed scientific truths, (as the laws of motion,) be *necessary* truths; and if they are necessary, (which I have attempted to shew that in a certain sense they are,) *on what ground* their necessity rests. These questions may be discussed in a general form, as I have elsewhere attempted to shew. But it may be instructive also to follow the general arguments into the form which they assume in special cases; and to exhibit, in a distinct shape, the incongruities into which the opposite false doctrine leads us, when applied to particular examples. This accordingly is what I propose to do in the present memoir, with regard to the proposition stated at the head of this paper, namely, that *all matter is heavy*.

At first sight it may appear a doctrine altogether untenable to assert that this proposition is a necessary truth: for, it may be urged, we have no difficulty in conceiving matter which is not heavy; so that matter without weight is a conception not inconsistent with itself; which it must be if the reverse were a necessary truth. It may be added, that the possibility of conceiving matter without weight was shewn in the controversy which ended in the downfall of the phlogiston theory of chemical composition; for some of the reasoners on this subject asserted phlogiston to be a body with positive levity instead of gravity, which hypothesis, however false, shews that such a supposition is possible. Again, it may be said that *weight* and *inertia* are two separate properties of matter: that mathematicians measure the quantity of matter by the inertia, and that we learn by experiment only that the weight is proportional to the inertia; Newton's experiments with pendulums of different materials having been made with this very object.

I proceed to reply to these arguments. And first, as to the possibility of conceiving matter without weight, and the argument thence deduced, that the universal gravity of matter is not a necessary truth, I remark, that it is indeed just, to say that we cannot even distinctly conceive the contrary of a necessary truth to be true; but that this impossibility can be asserted only of those perfectly distinct conceptions which result from a complete developement of the fundamental idea and its consequences. Till we reach this stage of developement, the obscurity and indistinctness may prevent our perceiving absolute contradictions, though they exist. We have abundant store of examples of this, even in geometry and arithmetic; where the truths are universally allowed to be necessary, and where the relations which are impossible, are also inconceivable, that is, not conceivable distinctly. Such relations, though not distinctly conceivable, still often appear conceivable and possible, owing to the indistinctness of our ideas. Who, at the first outset of his geometrical studies, sees any impossibility in supposing the side and the diagonal of a square to have a common measure? Yet they can be rigorously proved to be incommensurable, and therefore the attempt distinctly to conceive a common measure of them must fail. The attempts at the geometrical duplication of the cube, and the supposed solutions, (as that of Hobbes) have involved absolute contradictions; yet this has not prevented their being long and obstinately entertained by men, even of minds acute and clear in other respects. And the same might be shewn to be the case in arithmetic. It is plain, therefore, that we cannot, from the supposed possibility of conceiving matter without weight, infer that the contrary may not be a necessary truth.

Our power of judging, from the compatibility or incompatibility of our conceptions, whether certain propositions respecting the relations of ideas are true or not, must depend entirely, as I have said, upon the degree of developement which such ideas have undergone in our minds. Some of the relations of our conceptions on any subject are evident upon the first steady contemplation of the fundamental idea by a sound mind: these are the *axioms* of the subject. Other propositions may be deduced from the axioms by strict logical reasoning. These propositions are no less *necessary* than the axioms, though to common minds their *evidence* is very different. Yet as we become familiar with the steps by which these ulterior truths are deduced from the axioms, *their* truth also becomes evident, and the contrary becomes inconceivable. When a person has familiarized himself with the first twenty-six propositions of Euclid, and not till then, it becomes evident to him, that parallelograms on the same base and between the same parallels are equal; and he cannot even conceive the contrary. When he has a little further cultivated his geometrical powers, the equality of the square on the hypothenuse of a right-angled triangle to the squares on the sides, becomes also evident; the steps by which it is demonstrated being so familiar to the mind as to be apprehended without a conscious act. And thus, the contrary of a necessary truth cannot be distinctly conceived; but the incapacity of forming such a conception is a condition which depends upon cultivation, being intimately connected with the power of rapidly and clearly perceiving the connection of the necessary truth under consideration with the elementary principles on which it depends. And thus, again, it may be that there is an absolute impossibility of conceiving matter without weight; but then, this impossibility may not be apparent, till we have traced our fundamental conceptions of matter into some of their consequences.

The question then occurs, whether we can, by any steps of reasoning, point out an inconsistency in the conception of matter without weight. This I conceive we may do, and this I shall attempt to shew.

The general mode of stating the argument is this:—the quantity of matter is measured by those sensible properties of matter which undergo quantitative addition, subtraction and division, as the matter is added, subtracted and divided. The quantity of matter cannot be known in any other way. But this mode of measuring the quantity of matter, in order to be true at all, must be universally true. If it were only partially true. the limits within which it is to be applied would be arbitrary; and therefore the whole procedure would be arbitrary, and, as a method of obtaining philosophical truth, altogether futile.

We may unfold this argument further. Let the contrary be supposed, of that which we assert to be true: namely, let it be supposed that while all other kinds of matter are heavy, (and of course heavy in proportion to the quantity of matter) there is one kind of matter which is absolutely destitute of weight; as, for instance, phlogiston, or any other element.

Then where this *weightless* element (as we may term it) is mixed with *weighty* elements, we shall have a compound, in which the weight is no longer proportional to the quantity of matter. If, for example, 2 measures of heavy matter unite with 1 measure of phlogiston, the weight is as 2, and the quantity of matter as 3. In all such cases, therefore, the weight ceases to be the measure of the quantity of matter. And as the proportion of the weighty and the weightless matter may vary in innumerable degrees in such compounds, the weight affords no criterion at all of the quantity of matter in them. And the smallest admixture of the weightless element is sufficient to prevent the weight from being taken as the measure of the quantity of matter.

But on this hypothesis, how are we to distinguish such compounds from bodies consisting purely of heavy matter? How are we to satisfy ourselves that there is not, in every body, some admixture, small or great, of the weightless element? If we call this element *phlogiston*, how shall we know that the bodies with which we have to do are, any of them, absolutely free from phlogiston?

We cannot refer to the weight for any such assurance; for by supposition the presence and absence of phlogiston makes no difference in the weight. Nor can any other properties secure us at least from a very small admixture; for to assert that a mixture of 1 in 100 or 1 in 10 of phlogiston would always manifest itself in the properties of the body, must be an arbitrary procedure, till we have proved this assertion by experiment: and we cannot do this till we have learnt some mode of measuring the quantities of matter in bodies and parts of bodies; which is exactly what we question the possibility of, in the present hypothesis.

Thus, if we assume the existence of an element, *phlogiston*, devoid of weight, we cannot be sure that every body does not contain some portion of this element; while we see that if there be an admixture of such an element, the weight is no longer any criterion of the quantity of matter. And thus we have proved, that if there be any kind of matter which is not heavy, the weight can no longer avail us, *in any case or to any extent*, as a measure of the quantity of matter.

I may remark, that the same conclusion is easily extended to the case in which phlogiston is supposed to have absolute levity; for in that case, a certain mixture of phlogiston and of heavy matter would have no weight, and might be substituted for phlogiston in the preceding reasoning.

I may remark, also, that the same conclusion would follow by the same reasoning, if any kind of matter, instead of being void of weight, were heavy, indeed, but not *so* heavy, in proportion to its quantity of matter, as other kinds.

On all these hypotheses there would be no possibility of measuring quantity of matter by weight at all, in any case, or to any extent.

But it may be urged, that we have not yet reduced the hypothesis of matter without weight to a contradiction; for that mathematicians measure quantity of matter, not by weight, but by the other property, of which we have spoken, inertia.

To this I reply, that, practically speaking, quantity of matter is always measured by weight, both by mechanicians and chemists: and as we have proved that this procedure is utterly insecure in all cases, on the hypothesis of weightless matter, the practice rests upon a conviction that the hypothesis is false. And yet the practice is universal. Every experimenter measures quantity of matter by the balance. No one has ever thought of measuring quantity of matter by its inertia practically: no one has constructed a measure of quantity of matter in which the matter produces its indications of quantity by its motion. When we have to take into account the inertia of a body, we inquire what its weight is, and assume this as the measure of the inertia; but we never take the contrary course, and ascertain the inertia first in order to determine by that means the weight.

But it may be asked, Is it not then true, and an important scientific truth, that the *quantity of matter* is measured by the *inertia?* Is it not true, and proved by experiment, that the *weight* is *proportional* to the *inertia?* If this be not the result of Newton's experiments mentioned above, what, it may be demanded, do they prove?

To these questions I reply: It is true that quantity of matter is measured by the inertia, for it is true that inertia is as the quantity of matter.

This truth is indeed one of the laws of motion. That weight is proportional to inertia is proved by experiment, as far as the laws of motion are so proved: and Newton's experiments prove one of the laws of motion, so far as any experiments can prove them, or are needed to prove them.

That inertia is proportional to weight, is a law equivalent to that law which asserts, that when pressure produces motion in a given body, the velocity produced in a given time is as the pressure. For if the velocity be as the pressure, when the body is given, the velocity will be constant if the inertia also be as the pressure. For the inertia is understood to be that property of bodies to which, *ceteris paribus*, the velocity impressed is *inversely* proportional. One body has twice as much inertia as another, if, when the same force acts upon it for the same time, it acquires but half the velocity. This is the fundamental conception of *inertia*.

In Newton's pendulum experiments, the pressure producing motion was a certain resolved part of the weight, and was proportional to the weight. It appeared by the experiments, that whatever were the material of which the pendulum was formed, the rate of oscillation was the same; that is, the velocity acquired was the same. Hence the inertia of the different bodies must have been in each case as the weight: and thus this assertion is true of all different kinds of bodies.

Thus it appears that the assertion, that inertia is universally proportional to weight, is equivalent to the law of motion, that the velocity is as the pressure. The conception of inertia (of which, as we have said, the fundamental conception is, that the velocity impressed is inversely proportional to the inertia,) connects the two propositions so as to make them identical.

Hence our argument with regard to the universal gravity of matter brings us to the above law of motion, and is proved by Newton's experiments in the same sense in which that law of motion is so proved.

Perhaps some persons might conceive that the identity of weight and inertia is obvious at once; for both are merely resistance to motion;—inertia, resistance to all motion (or change of motion)—weight, resistance to motion upwards.

But there is a difference in these two kinds of resistance to motion. Inertia is instantaneous, weight is continuous resistance. Any momentary impulse which acts upon a free body overcomes its inertia, for it changes its motion; and this change once effected, the inertia opposes any return to the former condition, as well as any additional change. The inertia is thus overcome by a momentary force. But the weight can only be overcome by a continuous force like itself. If an impulse act in opposition to the weight, it may for a moment neutralize or overcome the weight; but if it be not continued, the weight resumes its effect, and restores the condition which existed before the impulse acted.

But weight not only produces rest, when it is resisted, but motion, when it is not resisted. Weight is measured by the reaction which would balance it; but when unbalanced, it produces motion, and the velocity of this motion increases constantly. Now what determines the velocity thus produced in a given time, or its rate of increase? What determines it to have one magnitude rather than another? To this we must evidently reply, *the inertia.* When weight produces motion, the inertia is the reaction which makes the motion determinate. The accumulated motion produced by the action of unbalanced weight is as determinate a condition as the equilibrium produced by balanced weight. In both cases the condition of the body acted on is determined by the opposition of the action and reaction.

Hence inertia is the reaction which opposes the weight, when unbalanced. But by the conception of action and reaction, (as mutually determining and determined,) they are measured by each other; and hence the inertia is necessarily proportional to the weight.

But when we have reached this conclusion, the original objection may be again urged against it. It may be said, that there must be some fallacy in this reasoning, for it proves a state of things to be necessary when we can so easily conceive a contrary state of things. Is it denied, the opponent may ask, that we can readily imagine a state of things in which bodies have no weight? Is not the uniform tendency of all bodies in the same direction not only not necessary, but not even true? For they do in reality tend, not with equal forces in parallel lines, but to a center with unequal forces, according to their position: and we can conceive these differences of intensity and direction in the force to be greater than they really are; and can with equal ease suppose the force to disappear altogether.

To this I reply, that certainly we may conceive the weight of bodies to vary in intensity and direction, and by an additional effort of imagination, may conceive the weight to vanish: but that in all these suppositions, even in the extreme one, we must suppose the rule to be universal. If *any* bodies have weight, *all* bodies must have weight. If the direction of weight be different in different points, this direction must still vary according to the *law of continuity;* and the same is true of the intensity of the weight. For if this were not so, the rest and motion, the velocity and direction, the permanence and change of bodies, as to their mechanical condition, would be arbitrary and incoherent: they would not be subject to mechanical ideas; that is, not to ideas at all: and hence these conditions of objects would in fact be inconceivable. In order that the universe may be possible, that is, may fall under the conditions of intelligible conceptions, we must be able to conceive a body at rest. But the rest of bodies (except in the absolute negation of all force) implies the equilibrium of opposite forces. And one of these opposite forces must be a *general* force, as weight, in order that the universe may be governed by general conditions. And this general force, by the conception of force, may produce motion, as well as equilibrium; and this motion again must be determined, and determined by general conditions; which cannot be, except the communication of motion be regulated by an inertia proportional to the weight.

But it will be asked, Is it then pretended that Newton's experiment, by which it was intended to prove inertia proportional to weight, does really prove nothing but what may be demonstrated *à priori?* Could we know, without experiment, that all bodies,—gold, iron, wood, cork,—have inertia proportional to their weight? And to this we reply, that experiment holds the same place in the establishment of this, as of the other fundamental doctrines of mechanics. Intercourse with the external world is requisite for developing our ideas; measurement of phenomena is needed to fix our conceptions and to render them precise: but the result of our experimental studies is, that we reach a position in which our convictions do not rest upon experiment. We learn by observation truths of which we afterwards see the necessity. This is the case with the laws of motion, as I have repeatedly endeavoured to shew. The same will appear to be the case with the proposition, that bodies of different kinds have their inertia proportional to their weight.

For bodies *of the same kind* have their inertia proportional to their weight, both quantities being proportional to the quantity of matter. And if we compress the same quantity of matter into half the space, neither the weight nor the inertia is altered, because these depend on the quantity of matter alone. But in this way we obtain a body of *twice the density;* and in the same manner we obtain a body of any other density. Therefore whatever be the density, the inertia is proportional to the quantity of matter. But the mechanical relations of bodies cannot depend upon any difference of *kind, except* a difference of density. For if we suppose any fundamental difference of mechanical nature in the particles or component elements of bodies, we are led to the same conclusion, of arbitrary, and therefore impossible, results, which we deduced from this supposition with regard to weight. Therefore all bodies of different density, and hence, all bodies whatever, must have their inertia proportional to their weight.

Hence we see, that the propositions, that all bodies are heavy, and that inertia is proportional to weight, necessarily follow from those fundamental ideas which we unavoidably employ in all attempts to reason concerning the mechanical relations of bodies. This conclusion may perhaps appear the more startling to many, because they have been accustomed to expect that fundamental ideas and their relations should be self-evident at our first contemplation of them. This, however, is far from being the case, as I have already shewn. It is not the *first,* but the most complete and developed condition of our conceptions which enables us to see what are axiomatic truths in each province of human speculation. Our fundamental ideas are necessary conditions of knowledge, universal forms of intuition, inherent types of mental developement; they may even be termed, if any one chooses, results of connate intellectual tendencies; but we cannot term them *innate* ideas, without calling up a large array of false opinions. For innate ideas were considered as capable of composition, but by no means of simplification: as most perfect in their original condition; as to be found, if any where, in the most uneducated and most uncultivated minds; as the same in all ages, nations, and stages of intellectual culture; as capable of being referred to at once, and made the basis of our reasonings, without any special acuteness or effort: in all which circumstances the Fundamental Ideas of which we have spoken, are opposed to Innate Ideas so understood.

I shall not, however, here prosecute this subject. I will only remark, that Fundamental Ideas, as we view them, are not only not innate, in any usual or useful sense, but they are not necessarily *ultimate* elements of our knowledge. They are the results of our analysis so far as we have yet prosecuted it; but they may themselves subsequently be analysed. It may hereafter appear, that what we have treated as different Fundamental Ideas have, in fact, a connexion, at some point below the structure which we erect upon them. For instance, we treat of the mechanical ideas of force, matter, and the like, as distinct from the idea of substance. Yet the principle of measuring the quantity of matter by its weight, which we have deduced from mechanical ideas, is applied to determine the substances which enter into the composition of bodies. The idea of substance supplies the axiom, that the whole quantity of matter of a compound body is equal to the sum of the quantities of matter of its elements. The mechanical ideas of force and matter lead us to infer that the quantity both of the whole and its parts must be measured by their weights. *Substance* may, for some purposes, be described as that to which properties belong; *matter* in like manner may be described as that which resists force. The former involves the Idea of permanent Being; the latter, the Idea of Causation. There may be some elevated point of view from which these ideas may be seen to run together. But even if this be so, it will by no means affect the validity of reasonings founded upon these notions, when duly determined and developed. If we once adopt a view of the nature of knowledge which makes necessary truth possible at all, we need be little embarrassed by finding how closely connected different necessary truths are; and how often, in exploring towards their roots, different branches appear to spring from the same stem.

W. WHEWELL.

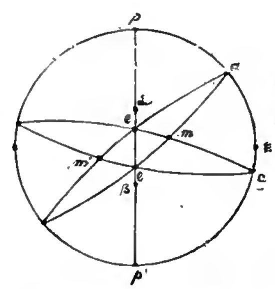
GRANGE,
Aug. 31, 1840.

XIII. *On the Position of the Axes of Optical Elasticity in Crystals belonging to the Oblique-Prismatic System. By* W. H. MILLER, M.A. F.R.S. *Fellow and Tutor of St. John's College, and Professor of Mineralogy in the University of Cambridge.*

[Read *March* 21, 1836.]

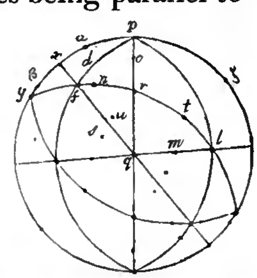
IN a Memoir printed in the 5th Volume of the Cambridge Transactions it is stated, that in crystals belonging to the Oblique-Prismatic System one of the three rectangular axes of optical elasticity was always found to coincide with that crystallographic axis (Y, Y′) which, in crystals of this system, is perpendicular to the other two: but that the positions of the other axes of optical elasticity ($\xi\xi'$, $\zeta\zeta'$) had no known relation to the form of the crystal. In some oblique-prismatic crystals, however, it was found that one of the axes of optical elasticity $\xi\xi'$, $\zeta\zeta'$ was also the axis of a principal zone. In the crystals which I have examined since the publication of the paper already alluded to, by the same method, this coincidence is found to occur less frequently. Upon the whole, however, there seems to be no reason for supposing it accidental in the instances (five or six out of twenty) in which it has been observed; but rather that it is a particular case of some general law connecting the form and optical properties of crystals, in the discovery of which it is hoped the observations here recorded may be in some degree instrumental.

The crystals selected for examination are taken principally from among those which have been described by Mr Brooke in the *Annals of Philosophy* for 1823 and 1824. The mutual inclination of two faces is expressed by the angle between their normals, or the angular distance of their "poles." An explanation of the notation in which the symbols of the simple forms are expressed, and of the method of representing the form of a crystal by its "sphere of projection," will be found in the *Cambridge Transactions*, Vol. V. p. 433. The velocity of light in air divided by its velocity within the crystal, for a ray in the plane of the optic axes, and polarized in the same plane, is denoted by μ. I being the refracting angle of a prism having its edge perpendicular to the plane of the optic axes, and D the minimum deviation of a ray refracted through it, polarized in the plane of the optic axes, $\mu \sin \frac{1}{2} I = \sin \frac{1}{2} (D + I)$. The index of refraction of the oil used in some of the observations is 1.4706 for the brightest rays of the spectrum. α, β; ζ, ξ, denote the extremities of radii of the sphere of projection drawn parallel to the optic axes and axes of optical elasticity respectively.



(1). In Oxalic Acid, $\ddot{\underline{C}}\dot{\underline{H}}^3$, the cleavages being parallel to the faces m, $mm' = 63^\circ.5'$, $ee' = 34^\circ.32$, $pa = 50^\circ.40'$, $cp' = 76^\circ.45'$, $ac = 52^\circ.35'$, $pm = 81^\circ.34'$, $am = 61^\circ.13',5$, $cm = 62^\circ.55',5$. The symbols of the simple forms are, p $\{0\,0\,1\}$, m $\{1\,1\,0\}$, e $\{0\,1\,1\}$, a $\{1\,0\,1\}$, c $\{1\,0\,\bar{1}\}$.

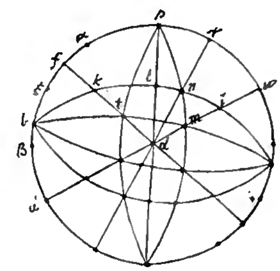
The apparent directions of the optic axes seen in oil through the faces p lie in a plane perpendicular to the faces p, e, and make with each other an angle of $115^\circ.30'$. $\mu = 1.499$. Hence $\alpha\beta = 68^\circ$, and the axis of optical elasticity ξ coincides with the axis of the zone $pee'p'$.



(2). In Sphene, the faces being denoted by the same letters as in the treatises of Mohs and Naumann, and the principal cleavages being parallel to the faces l, y, $ql = 66^\circ.54'$, $yl = 131^\circ.21'$, $lp = 85^\circ.33'$, $yx = 21^\circ.5'$, $xp = 39^\circ.19'$, $pql = 85^\circ.10'$, $pqt = 53^\circ.36'$, $pqn = 28^\circ.6'$. The symbols of the simple forms are, q $\{0\,1\,0\}$, p $\{0\,0\,1\}$, l $\{1\,1\,0\}$, m $\{1\,3\,0\}$, r $\{0\,1\,1\}$, y $\{\bar{1}\,0\,1\}$, x $\{\bar{1}\,0\,2\}$, o $\{0\,1\,3\}$, t $\{1\,2\,1\}$, d $\{\bar{1}\,1\,3\}$, n $\{\bar{1}\,2\,3\}$, u $\{\bar{1}\,6\,3\}$, f $\{\bar{1}\,1\,2\}$, s $\{\bar{1}\,4\,1\}$.

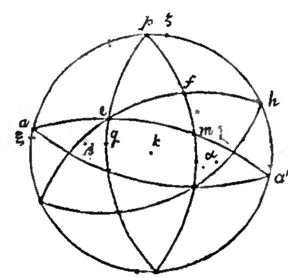
The apparent directions of the optic axes seen in water through the faces x lie in a plane perpendicular to the faces xp, and make angles of about 18°. 40′, with a normal to the face x. $\mu = 1.631$. Hence $\alpha\beta = 30°.22'$, and the axis of elasticity ζ coincides with the axis of the zone xfq.

(3). In Phosphate of Soda, $Na^2 \ddot{\bar{P}} \dot{\bar{H}}^{25}$, according to Mitscherlich (*Annales de Chimie*, Tome 19.) $ur = 33°.8'$, $rp = 25°.24'$, $pf = 50°.48'$, $fb = 33°.25'$, $bu' = 37°.17'$, $dm = 33°.55'$, $di = 65°.4'$, $dn = 53°.12'$, $dl = 52°.9'$, $dt = 36°.30'$, $dk = 67°.6'$, $pt = 67°.55'$, $pn = 31°.30'$, $pm = 73°.3'$. The symbols of the simple forms are, u $\{1\,0\,0\}$, d $\{0\,1\,0\}$, p $\{0\,0\,1\}$, m $\{1\,1\,0\}$, n $\{1\,1\,1\}$, t $\{\bar{1}\,1\,1\}$, b $\{\bar{2}\,0\,1\}$, f $\{\bar{1}\,0\,1\}$, r $\{1\,0\,1\}$, l $\{0\,2\,3\}$, k $\{\bar{3}\,1\,3\}$, i $\{3\,1\,0\}$.



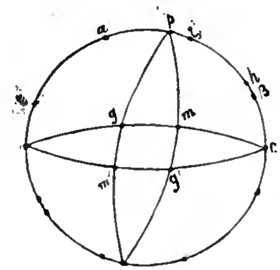
The optic axes lie in a plane perpendicular to the faces u, p, f. When the crystal is immersed in oil, the apparent direction of the optic axis α seen through the faces p makes an angle of 34°. 30′ with a normal to p, and an angle of 58°. 40′ with the apparent direction of the optic axis β seen through artificial surfaces nearly perpendicular to the optic axis β. $\mu = 1.40$ nearly. Hence $p\alpha = 36°.30'$, $p\beta = 93°.10'$, $p\xi = 64°.50'$. Therefore the axis of elasticity ξ very nearly coincides with the axis of the zone rnd. It is not possible to determine the positions of the optic axes of phosphate of soda very accurately on account of its feeble double refraction, the imperfection of its surfaces, and its tendency to effloresce.

(4). In Acetate of Soda, $\dot{Na}\bar{A}\dot{H}^6$, $ap = 76°.25'$, $ph = 68°.16'$, $ha' = 35°.15'$, $mk = 42°.15'$, $pm = 75°.35'$, $pf = 42°.43$, $pe = 60°.22'$, $pg = 81°.8',5$. The symbols of the simple forms are, h $\{1\,0\,0\}$, k $\{0\,1\,0\}$, p $\{0\,0\,1\}$, a $\{\bar{2}\,0\,1\}$, f $\{1\,1\,1\}$, e $\{\bar{1}\,1\,1\}$, g $\{\bar{2}\,2\,1\}$.



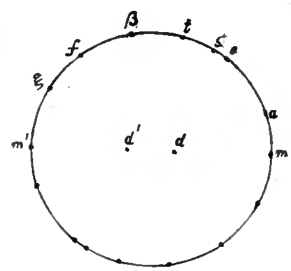
When the crystal is immersed in oil, the apparent directions of the optic axes seen through a slice bounded by artificial surfaces nearly parallel to the faces a, make with each other an angle of 62°. 30′; and the apparent direction of the optic axis β seen through artificial sections nearly perpendicular to β, makes an angle of 80°.30′, with a normal to p. $\mu = 1.464$. Hence $\alpha\beta = 117°.10'$, $p\zeta = 11°.9'$, $a\xi = 2°.26'$.

(5). In Acetate of Oxide of Zinc, $\dot{Zn}\overline{A}\dot{\underline{H}}^3$, the cleavage being parallel to the face p, $ph = 46^\circ.30'$, $pc = 79^\circ.55'$, $mm' = 67^\circ.24'$, $pm = 67^\circ.33'$, $pg = 75^\circ.30'$, $gg' = 58^\circ.43'$. The symbols of the simple forms are, c {1 0 0}, p {0 0 1}, h {1 0 1}, m {1 1 1}, g {$\bar{1}$ 1 1}.



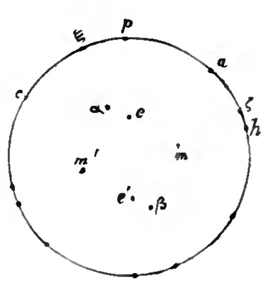
The optic axes lie in a plane perpendicular to the faces p, h, c. The apparent direction of the optic axis a, seen in air through the faces p, makes an angle of $50^\circ.15'$, with a normal to p. When the crystal is immersed in oil, the apparent directions of the optic axes seen through the faces p, make with each other an angle of $79^\circ.15'$. $\mu = 1.494$. Hence $p\zeta = 11^\circ.16'$, $h\zeta = 35^\circ.14'$, $a\beta = 84^\circ.30'$.

(6). In Bicarbonate of Potash, $\dot{K}\ddot{C}^2\dot{\underline{H}}$, $me = 53^\circ.15'$, $mt = 76^\circ.35'$, $mf = 127^\circ.52'$, $dd' = 42^\circ.0'$. The symbols of the simple forms are, m {1 0 0}, t {0 0 1}, f {$\bar{1}$ 0 1}, e {2 0 3}, d {1 1 0}.



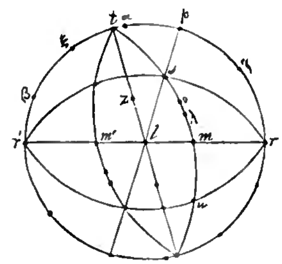
The apparent direction of the optic axis a seen in air through the faces e, makes an angle of $56^\circ.45'$, with a normal to e. The apparent directions of the optic axes seen in oil through the faces e make with each other an angle of 83°. $\mu = 1.482$. Hence $ea = 48^\circ.21'$, $e\beta = 47^\circ.53'$, $e\zeta = 6^\circ.28'$, $a\beta = 81^\circ.38'$.

(7). In Tartaric Acid, $\overline{T}\dot{\underline{H}}$, $mm' = 88^\circ.30'$, $ee' = 76^\circ.30'$, $pm = 97^\circ.10'$, $ph = 80^\circ.3'$. The symbols of the simple forms are, h {1 0 0}, p {0 0 1}, m {1 1 0}, a {1 0 1}, c {$\bar{1}$ 0 1}, e {0 1 1}.



The apparent directions of the optic axes seen in oil through artificial sections perpendicular to the faces p, h, lie in a plane inclined at an angle of $69^\circ.30'$ to the face p, and make with each other an angle of 103°. $\mu = 1.542$ nearly. Hence $p\xi = 20^\circ.30'$, $h\zeta = 10^\circ.33'$, $a\beta = 96^\circ.36'$.

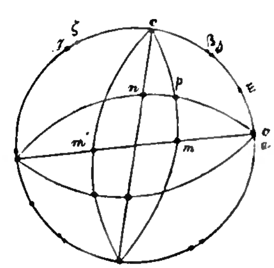
(8). In Pyroxene, the faces being denoted by the same letters as in the treatises of Mohs and Naumann, the symbols of the simple forms are, *p* {0 0 1}, *l* {0 1 0}, *r* {1 0 0}, *m* {1 1 0}, *f* {3 1 0}, *t* {$\bar{1}$ 0 1}, *s* {0 1 1}, *u* {$\bar{2}$ 1 1}, *z* {$\bar{1}$ 2 1}, *o* 1 2 1}, λ {2 3 1}.



The optic axes lie in a plane parallel to the face *l*. The apparent direction of the optic axis *α* seen in air through sections perpendicular to the faces *m*, *m'*, makes an angle of 74° with a normal to *r'*. The apparent direction of the optic axis *β* seen in water through the faces *r*, *r'*, makes an angle of 27°, 40′, with a normal to *r*. $\mu = 1.680$. Hence $\alpha r' = 80°.34'$, $\beta r' = 21°.38'$, $\xi r' = 56°.6'$.

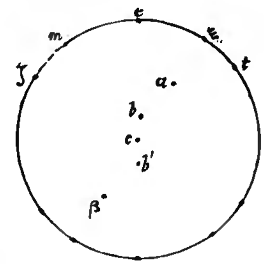
The crystals of Pyroxene in which I first attempted to determine the position of the optic axes were all twins composed of individuals of unequal size, the twin-axis being perpendicular to the face *r*. Consequently, a slice bounded by planes perpendicular to the faces *m*, *m'* exhibited two systems of rings unequally bright, making with each other an angle of 32°, which was bisected by the axis of the zone *mr*. These rings were erroneously supposed to belong to the same crystal, till the mistake was pointed out to me by Professor Nörrenberg. The best crystals which I have been able to procure for measurement give *pr* = 74°.20′, nearly. In a twin crystal in Mr Brooke's collection, the face *t* of one individual coincides accurately with the face *p* of the other. This shews that Pyroxene may quantitatively be referred to the right prismatic system. The position of the optic axes, as well as the nature of the symmetry of the faces *u*, *z*, *o*, λ, prove, however, clearly that it belongs to the oblique-prismatic system.

(9). In crystals of Sugar the faces are too uneven to admit of determining the angles they make with each other nearer than within, perhaps, half a degree of the truth. *a* being the face parallel to which a very distinct cleavage exists, *mm'* = 79°.20′, *ar* = 116°.40′, *ac* = 75°.30′, nearly. The symbols of the simple forms are, *a* {1 0 0}, *c* {0 0 1}, *r* {$\bar{1}$ 0 1}, *s* {1 0 1}, *n* {0 1 1}, *p* {1 1 1}.



The apparent directions of the optic axes α, β, seen in oil through the faces a, a', make angles of $1^\circ.32'$ and $49^\circ.58'$, with a normal to a. $\mu = 1.57$. Hence $a\alpha = 1^\circ.26'$, $a\beta = 45^\circ.50'$, $a\xi = 22^\circ.12'$, $r\zeta = 4^\circ.28'$.

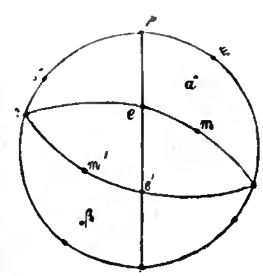
(10). In Tartrate of Potash, $\dot{K}\bar{T}\dot{H}^2$, the cleavages being parallel to m, t, $me = 37^\circ.47'$, $et = 52^\circ.42'$, $bb' = 45^\circ.20'$. The symbols of the simple forms are, c $\{0\ 1\ 0\}$, e $\{0\ 0\ 1\}$, t $\{1\ 0\ 1\}$, m $\{\bar{1}\ 0\ 1\}$, b $\{0\ 1\ 1\}$.



The apparent directions of the optic axes seen in oil through the faces t lie in a plane perpendicular to the face c, making an angle of $67^\circ.30'$ with the face t. They make with each other an angle of $64^\circ.45'$, and therefore angles of $38^\circ.43'$, with a normal to t. $\mu = 1.526$ nearly. Hence, supposing a ray in the direction of the optic axes to be refracted in the same manner as at the surface of glass, having 1.526 for its index of refraction, $t\xi = 21^\circ.20'$, $\alpha\beta = 118^\circ$ nearly.

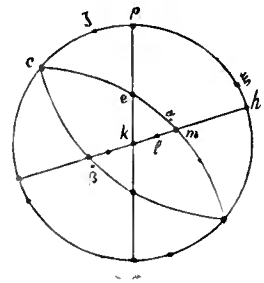
The above assumption, though not strictly correct, will not occasion any considerable error in the present instance. This appears to be the only practicable method of determining (approximately) the positions of the optic axes, when the plane in which they lie is not perpendicular to the faces through which they are seen. It is used in the two following cases.

(11). In Chlorate of Potash, $\dot{K}\dddot{\underline{Cl}}$, the cleavages being parallel to the faces m, m', $m\,m' = 104^\circ.0'$, $e\,e' = 79^\circ.30'$, $p\,m = 74^\circ.30'$. The symbols of the simple forms are, p $\{0\ 0\ 1\}$, m $\{1\ 1\ 0\}$, e $\{0\ 1\ 1\}$, c $\{\bar{1}\ 0\ 1\}$.



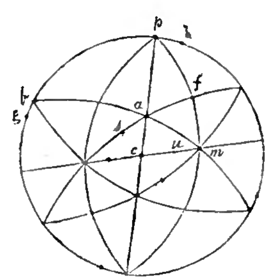
The apparent directions of the optic axes seen in oil through the faces p lie in a plane parallel to the axis of the zone pc, making an angle of 52° with the face p, and they make with each other an angle of $28^\circ.15'$. $\mu = 1.507$ nearly. Hence $p\,\xi = 37^\circ.42'$, $\alpha\beta = 152^\circ.30'$ nearly.

(12). In Sulphate of Soda, $\dot{N}a\ \dddot{S}u\ \underline{\dot{H}}^{10}$, $hp = 72^\circ.16'$, $pc = 49^\circ.15'$ $km = 40^\circ.12'$, $kl = 22^\circ.54'$, $ke = 49^\circ.54'$. The symbols of the simple forms are, k {0 1 0}, h {1 0 0}, p {0 0 1}, l {1 2 0}, e {0 1 1}, m {1 1 0}.



The apparent directions of the optic axes seen in oil through the faces h lie in a plane making an angle of 78°. 30′ with the face h, and make with each other an angle of 97°. 30′. $\mu = 1.44$ nearly. Hence $h\xi = 12^\circ.24'$, $\alpha\beta = 80^\circ.26'$ nearly.

(13.) In Hydrous Oxalate of Lime, $\dot{C}a\ \underline{\ddot{C}}\ \underline{\dot{H}}$, a new mineral species described by Mr Brooke in the Philosophical Magazine for June 1840, b being the face parallel to which a very distinct cleavage exists, $cm = 50^\circ.18'$, $cf = 65^\circ.28$, $ca = 37^\circ.24',5$, $cu = 31^\circ.3'$, $cs = 28^\circ.41'$, $pm = 76^\circ.46'$, $pb = 70^\circ.33'$, $pcm = 72^\circ.41'$. The symbols of the simple forms are, c {0 1 0}, p {0 0 1}, m {1 1 0}, a {0 1 1}, b {1 0 $\bar{1}$}, u {1 2 0}, f {1 1 2}, s {$\bar{1}$ 3 2}.



The optic axes could not be seen; the position of the axes of elasticity was however determined approximately by placing the crystal in a polarizing apparatus, having the planes of polarization and analyzation at right angles to each other, with the face c perpendicular to the axis of the instrument, and observing the position of the face p when the crystal ceased to transmit light. In this manner it was found that $b\xi = 8^\circ$.

W. H. MILLER.

XIV. *On a New Construction of the Going-Fusee.* *By* G. B. Airy, Esq. *Astronomer Royal.*

[Read March 2, 1840.]

I should not have presumed to occupy the time of the Cambridge Philosophical Society with a mere description of a mechanical construction, if I did not conceive that it might possess some interest for them, first, as a modification of or rather a substitute for a contrivance, whose elegance and importance have been universally acknowledged, but which fails (from practical reasons only) in certain cases. And secondly, as an object of local interest, the only existing application of the new construction being in the mounting of the magnificent telescope, which the University owes to the munificence of the Duke of Northumberland.

The object of the going-fusee is, as is well known, to maintain exactly the same action (whatever its amount may be) upon the first wheel of a clock, while the clock is being wound up, as while it is going in its ordinary way: supposing that the time required for winding up is not very long.

It was invented by Harrison; and has always appeared to me one of the most beautiful of the many beautiful contrivances in a highly-finished time-keeper.

When I was arranging the clock-work for the Northumberland telescope, I soon perceived that it would be necessary to depart from Harrison's construction in the going-fusee part. This was rendered imperative by the magnitude of the force which, as it appeared probable, would be required to maintain the motion of the clock. A strain of 100lbs. on the cord was to be provided for: and therefore the remontoir spring must be strong enough to support 100lbs. without breaking, yet sufficiently flexible and elastic to expand without great diminution of that force through a sensible space. There is doubtless no difficulty in satisfying these conditions in the case of a coach-spring, or where there is abundance of room: but, where the spring must be contained within not a clock but a clock-wheel, there is considerable difficulty. The only way in which I could hope to use the principle, must be by adopting a barrel, ratchet, click, and first wheel, exactly as in a kitchen-clock; and removing the ratchet-wheel of Harrison's fusee with its click and going-spring to the spindle of the next wheel, where the forces are much diminished. But here it would be necessary to use a spring which is coiled several times round the spindle; else, as this second wheel revolves more rapidly than the first, the spring would be too much relaxed before the cessation of the pressure of the hand allowed the weight to act again. The difficulty of manufacturing the spring would be great, and in all contrivances requiring a steel spring there was the risk of rust, against which I could not hope to secure the machinery.

I might have adopted the contrivance known as the endless cord of Huyghens, which has been employed in Fraunhofer's clocks. The only objection to the use of this construction for the Northumberland clock was, that the spikes in the gorge of the pulley, which are necessary to prevent the cord from slipping, would speedily have torn the cord to pieces, when a weight of 100 lb. was attached.

Abandoning the spring and the endless rope, my first idea was, to use a new weight in such a manner as to produce exactly the same effect and in the same place as Harrison's going-spring. Various constructions presented themselves; but those founded on the following principle, appeared the most feasible:—The action of a spring may be exactly imitated by that of a jointed lozenge: the two parts which are to be connected by the spring being two opposite angles of the lozenge, and the two other angles being pulled apart by the action of constant weights. In the application of this principle, the parts to be connected by the spring or lozenge would be on the circumference of the barrel and wheel, and the two other angles would therefore be on the same circumference: but there was no difficulty in effecting the pulling apart of these angles by a force in the axis of the barrel, which, by a proper application of bell-cranks, could easily be effected by a weight. But the construction produced a trifling friction in the ordinary going of the clock, and was not elegant, and I therefore abandoned it.

Finally, I had the good fortune to imagine a construction entirely different, with which, in all respects, I am fully satisfied. It is based upon the following principle. If a lever *abc*, fig. 1, is used to produce pressure at the point *c*, *b* being its fulcrum, and *a* the point at which the force is applied: then the same effect may be produced on *c*, by making *a* the fulcrum, provided that at *b* we apply a force exactly equal and opposite to the pressure which the fulcrum at *b* sustained when the force was applied at *a*.

To apply this to the first wheel of a clock. Suppose (for simplicity of present consideration and of future construction) the axes of the first wheel and second wheel to be in the same horizontal plane. Let *a* fig. 2. be the point of the barrel from which the weight depends: *b* its center, *c* the point at which the toothed wheel, connected with the barrel, acts upon the pinion of the next wheel. Then, during the ordinary action of the clock, *abc* may be considered as a lever, of which *b* is the fulcrum, sustaining a downwards pressure, *a* the point of application of the force, and *c* the point on which it is to produce an effect. Suppose, in the operation of winding up, the force acting at *a* to be removed. Then, by the theorem which I have lately mentioned, the same effect may still be produced on *c*; provided that we can so arrange our mechanism that *a* shall, during the operation of winding up, become the fulcrum; and that a force shall act upwards at *b*, exactly equal to the downwards pressure which *b* sustained during the clock's ordinary motion, the point *b* being not fixed (as before) but moveable. The mechanism necessary for this purpose is extremely simple.

Instead of supposing the pivot *b* of the barrel to turn in a hole in the clock-plate, let it turn in a hole in the arm of a frame, fig. 3, of which one side is *bad*, and which is itself a lever whose fulcrum projected in *a* is the line joining two pins turning in holes of the clock-plate, corresponding exactly in position to the point of the barrel from which the weight *W* depends. Suppose another weight *w* to be suspended from *d*, of such magnitude that it will exactly support (acting with the fulcrum *a*) the pressure which the lever-frame sustains at *b*. It will readily be remarked, that if the lever-frame be bent, as shewn in the figure, no nice adjustment of the weight of *w* is necessary. For, if the weight of *w* be a little too small, the preponderance of the pressure at *b* will depress *b*, and will thereby throw *d* so far in the horizontal direction that the increased power of the lever *ad* will enable the same weight *w* to balance the pressure at *b*. In like manner, if the weight *w* be somewhat too large, the approach of *d* in the horizontal direction, towards the vertical passing through *a*, will diminish its statical momentum, and thereby restore the equilibrium. The effect of either of these changes on the action of the wheel-teeth is to withdraw them from the teeth of the pinion by an almost infinitesimal quantity, of which the effect in practice is wholly insignificant.

We may therefore now be assured that we have provided a force acting upwards at *b*, exactly equal and opposite to the pressure which the fulcrum at *b* sustains during the ordinary motion of the barrel (inasmuch as our force does veritably support that pressure during the ordinary motion of the barrel), and competent to act with insignificant diminution of amount even if *b* be moved. One condition therefore of the change of lever-action is entirely satisfied. The other condition requires that the point *a* of the toothed-wheel shall be made, during the process of winding up, the fulcrum upon which the toothed-wheel turns for the time. But the corresponding point *a* of the lever-frame is the fulcrum upon which the lever-frame is able to turn for the time. Consequently all that is necessary to satisfy this condition is, to contrive that, during the process of winding up, the toothed-wheel shall be so connected with the lever-frame that it shall have absolutely the same motion which the lever-frame has on the fulcrum *a*; or, in other words, that the toothed-wheel and lever-frame shall (for the time) move *all in one piece*. All that is requisite for this purpose is, to make a ratchet in the inside of the ring of the toothed-wheel; and to make a click *f* to fall in the teeth of the ratchet, its center of motion being some convenient point *g* of the lever-frame. For then, upon taking off the pressure of *W* and the consequent pressure downwards on *b*, the pressure of *w* will immediately depress d till the end of f is firmly lodged upon a tooth of the ratchet: and then, inasmuch as the toothed-wheel is carried by the lever-frame at its center b and is thrust by the click f connected with the lever-frame, the continued descent of d will carry the toothed-wheel in just the same manner as if it were part of the lever-frame; and therefore the toothed-wheel will for the time revolve about a.

The two conditions therefore, which are required in the change of forces acting on the lever $a\ b\ c$ are entirely satisfied; and therefore the pressure at c during the winding up of the clock will be the same as during the ordinary going of the clock.

The following description of the movement may perhaps facilitate the understanding of the action of this mechanism.

While the clock is going in its ordinary way, the weight W descends, turning the barrel and wheel in such a direction that the teeth of the internal ratchet glide under the click f without producing on it or sustaining from it any effect. The action of the weight W and the resistance of the pinion at c produce a certain pressure on the lever-frame at b which causes the end d to assume a determinate position, in which it remains without motion so long as the weight W acts.

As soon as the pressure of W is relieved, the pressure on b ceases; the weight w preponderates, the end d drops, the end of f is thrust firmly against a tooth of the ratchet, and the continued action of w causes the toothed wheel to turn in a piece with the lever-frame round the center a, and thereby to produce a pressure at c, which, if a correspond exactly to the place at which W acted on the barrel, is exactly equal to the pressure which formerly actd at c.

If the action of W be suspended for a long time, the continued descent of d will bring d nearer to the vertical passing through a, and will thereby diminish the statical moment of w, and consequently will diminish the pressure at c. In this regard the action of this mechanism is exactly similar to that of the going-spring in Harrison's going-fusee.

One important point to which I have not yet alluded is the manner of winding up. It has been supposed all along that the act of winding up simply relieves the barrel from the pressure of W. This cannot be done by a square and winch upon the axis b in the usual way. For the action of the hand in winding up would then produce a force which may be resolved into a couple acting on the barrel and a force of variable direction acting at b: which differs entirely from our supposed relief of the pressure of W. But it can be done easily by inseparably attaching a toothed-wheel h to the barrel, and mounting a toothed-wheel k with its centre of motion on the clock-plate, so that the center of k shall be in the same horizontal line with a, and that the teeth of k may work in those of h: the winding-up-key being applied to the axis of k. For then the act of turning k produces no effect on the barrel except a pressure upwards at the very point where the weight of W produces a pressure downwards (any incidental pressure in the direction of a radius of the barrel, arising from the slope of the surface of the teeth, evidently having no effect on the angular motion about a). And therefore, as that pressure upwards must necessarily be equal in magnitude to the pressure produced by W, it follows that we may consider the pressure of W as simply relieved in this way of winding up the clock. The wheel k, it is to be observed, may be of any size or any number of teeth whatever.

The going-fusee is now complete in its action, so far as regards the use of a determinate weight W. But by a trifling alteration it will be made perfect for any weight whatever, without requiring any other change when the weight W is changed.

Suppose the lever-frame to be so loaded at d that the lever-frame when carrying the barrel and toothed-wheel may be nearly in equilibrium about a. Then the weight w must be in a constant proportion to W. Now it will be possible always to arrange the suspension of a single weight by a line with pulleys attached to the barrel and to d, so that the tension of the line acting on d shall be to the tension of that acting upon the barrel in the constant proportion which may be assumed.

Consequently the action of that single weight, whatever be its amount, will then produce two forces such as are proper for the action of this going-fusee: and therefore upon any change in that weight there will still be two forces such as are proper for that action; and an alteration in the suspended weight therefore will not require any other alteration in the adjustments of the mechanism.

If it be required, for instance, that the forces corresponding to W and w shall be equal, we must adopt the construction represented in figure 4, which, for its simplicity, may be considered preferable to any other. If it be required that the force corresponding to w shall be double that corresponding to W, we must adopt the construction of figure 5. This is the construction adopted by the mechanic who (under my general direction) constructed the clock-work of the Northumberland Telescope. The wheels h and k are, for clearness, omitted in figures 4 and 5.

The length and inclination of the arm ad will depend upon the horizontal distance between the verticals through a and d: and this horizontal distance will be found by such a calculation as the following. Suppose, (for instance) the diameter of the barrel to be half that of the toothed-wheel. The force W acting on the barrel will produce a force $\frac{W}{2}$ at c, and the pressure at b will therefore be $\frac{3W}{2}$. This pressure acting on the arm ba of the lever whose fulcrum is a, is to be balanced by the pressure w acting at d: or $\frac{3W}{2} \times ba = w \times al$. Consequently $al = \frac{W}{w} \times \frac{3}{2} ba$. Thus in the instance of fig. 4, where $\frac{W}{w} = 1$, al must $= \frac{3}{2} ba$; in the instance of fig. 5, where $\frac{W}{w} = \frac{1}{2}$, al must $= \frac{3}{4} ba$. In determining the inclination to be adopted for the arm ad the mechanic must be guided only by the following considerations: that if ad be nearly horizontal, the failing of power in the action of the going-fusee (similar to the weakening of a spring by expansion) will be small, but the angle through which the lever-frame must turn, in order to correct any small error of adjustment, will be large: whereas, if ad be greatly inclined to the horizon, the action of the going-fusee fails rapidly during the suspension of the action of W, but a small error of adjustment is corrected by a small motion of the lever-frame. I should think that an inclination of 40° to the horizon would be found convenient.

For fully understanding the action of the mechanism, the following remarks may be useful.

If the wheel *k*, fig. 3, be forced a little in the direction opposite to that of winding up, the clock will go for some time without any descent of *W*. For, (using *a* to denote the point of the barrel where the teeth of *k* act on those of *h*), *abc* may then be considered as a lever whose fulcrum is *c*: the force acting downwards on *a*, will depress *b*, and will make several teeth of the internal ratchet glide under *f*, and will, at the same time, carry *d* further in the horizontal direction; then if the force on *k* ceases, the force *w*, acting now with increased statical momentum, will thrust *f* against the teeth of the ratchet, and will maintain the pressure and motion at *c*, by the motion of the whole lever-frame and toothed wheel round *a*. In this respect, the action of this mechanism is similar to that of Harrison's going-fusee.

If the distance *ba* (using *a* to denote the fulcrum of the lever-frame) be greater than the radius of the barrel, the force which acts on *c* during the winding up, is greater than that which acts during the ordinary going of the clock. If *ba* be less than the radius of the barrel, the force which acts during the winding up is less.

It has been supposed in the whole of this explanation, that *b* and *c* are in the same horizontal line, and that the pressure which the teeth of the wheel exert on these of the pinion is to be upwards. If the pressure is to be downwards, the only difference in the form of the construction will be, that the lever-pivot *a* will be between the wheel-center and the pinion-center, and that the inclined arm *ad* of the lever-frame will be turned towards the pinion; its length, &c. will be determined by the same considerations as in the case which has been fully treated. If *b* and *c* are not in the same horizontal line, all that is necessary is, to make the barrel-line pass over a pulley, so that the direction of its action shall be perpendicular to the line *bc*: no alteration whatever is needed for the arm *ad*, or the line which acts on it. An instance of such a case is represented in figure 6.

In all cases, the center of the pinion, the center of the toothed-wheel, the pivot of the lever-frame, and the center of the winding-up-wheel, must be in the same straight line: and the pivot of the lever-frame, the place at which the cord is a tangent to the barrel, and the place at which the teeth of the winding-up-wheel act on those of the barrel-wheel, must (as projected on the clock-plate) coincide.

I shall terminate this paper by referring to the two isometrical drawings of the new going-fusee, in figures 7 and 8. The first, fig. 7, represents the lever-frame with the barrel, toothed-wheel, internal ratchet, and clicks, as viewed from the side on which the clock is wound up. The clock-plate is supposed to be taken off: and, as the winding-up-wheel *k* is carried by the clock-plate, the support of that wheel is not represented. The second, fig. 8, represents the whole of the mechanism, as viewed from the side opposite to that on which the clock is wound up: the clock-plate opposite to the winding-up-side being taken off.

G. B. AIRY.

ROYAL OBSERVATORY, GREENWICH,
Feb. 5, 1840.

Fig. 3.

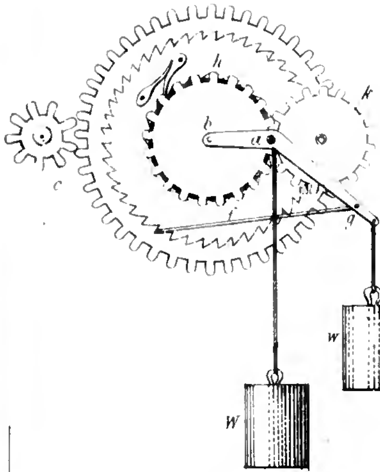


Fig. 4.

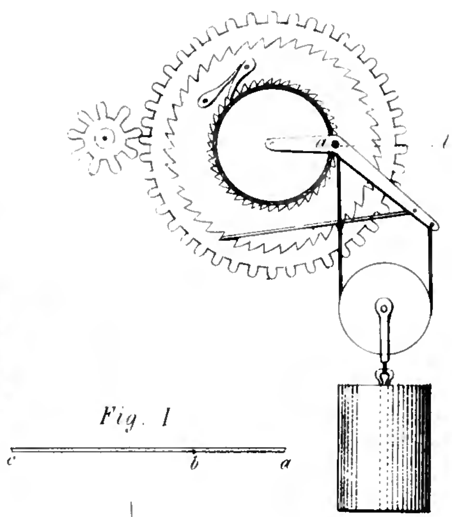


Fig. 5.

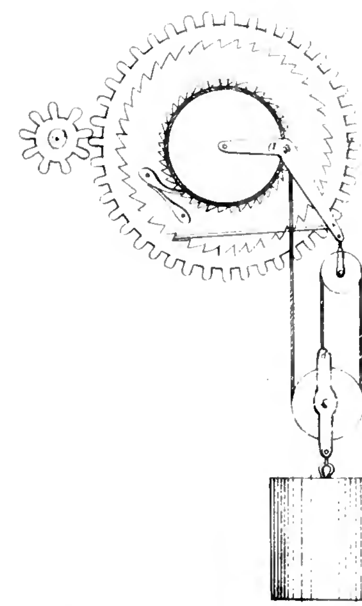
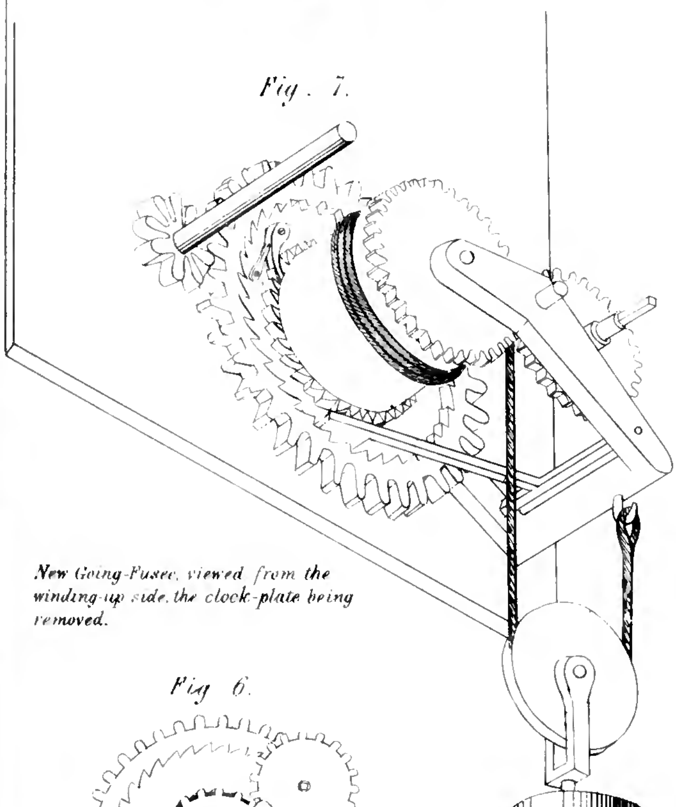


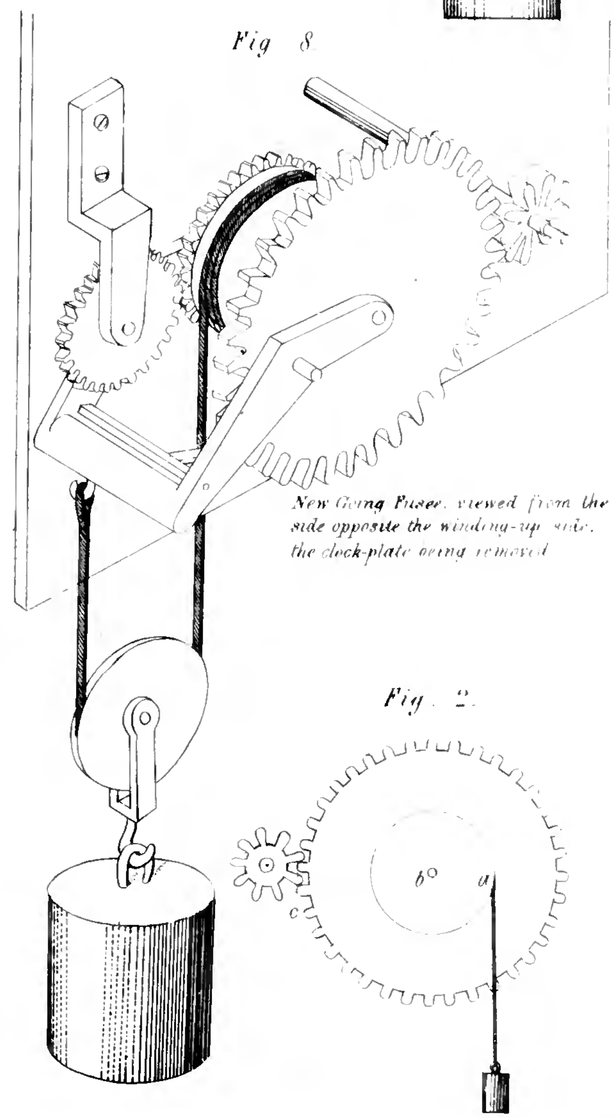
Fig. 1.

Fig. 7.



New Going-Fusee, viewed from the winding-up side, the clock-plate being removed.

Fig. 8.



New Going Fusee, viewed from the side opposite the winding-up side, the clock-plate being removed.

Fig. 6.

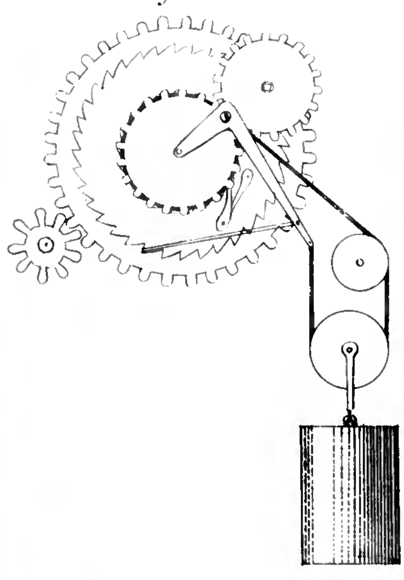


Fig. 2.

TRANSACTIONS

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XV. *On Spurious Rainbows.* *By* W. H. MILLER, M.A. F.R.S. *Professor of Mineralogy in the University of Cambridge.*

[Read *March* 22, 1841.]

THE sixth volume of the Transactions of the Cambridge Philosophical Society contains a Memoir by the Astronomer Royal, on the Intensity of Light in the neighbourhood of a Caustic, in which the relative distances of the brightest parts of the first spurious bow, and of the first and second dark rings, from the geometrical place of the bow, are determined by calculations founded on the undulatory theory. The numbers to which he finds these distances proportional are,

Brightest part of bow	1.08
Dark ring between the bow and the first spurious bow	2.48
Brightest part of the first spurious bow	3.47
Dark ring between the first and second spurious bows	4.4 (probably).

It appears also, that the illumination extends beyond the place of the geometrical bow, the intensity of the light there being about 0.442 of the intensity at the point of maximum brightness.

In order to compare these results with observation, I employed the method of exhibiting rainbows and the accompanying spurious bows, invented by M. Babinet (Poggendorff's Annalen, B. XLI. S. 139). When a beam of light admitted horizontally through a narrow vertical slit falls upon a vertical cylindrical stream of water, portions of the primary and secondary rainbows and of a large number of the spurious bows may be seen either with the naked eye or through a telescope, forming a series of vertical coloured bands, arranged in a horizontal line to the right and left of the point opposite to that from which the light is transmitted. A graduated circle placed horizontally with its center in the axis of the cylindrical stream, carrying a small telescope parallel to its plane, and having its object-end about one inch distant from the axis of the circle, served to measure the angle between the line of light and any one of the luminous bars.

The diameter of the stream was determined in the following manner. A lens of about 0.77 inch focal length was placed between the object-glass of the telescope and the stream, at the distance of its focal length from the axis of the latter, and the angle which the diameter of the stream subtended when seen through the lens, measured. Next, a scale of millimetres divided on glass, being placed in the focus of the lens, the angle subtended by two lines distant one millimetre from each other was measured. From these two angles the diameter of the stream may be readily calculated.

In the first observations the diameter of the stream was about 0.022 inch, and the light used was that of the Sun. The mixture of different colours rendered it very difficult to fix upon the brightest parts of the bars, especially of those corresponding to the principal bows.

The mean of eight observations of the primary and two of the secondary gave,

	° ′
Radius of the brightest part of the primary bow	41.32
Radius of the brightest part of its first spurious bow	40.27
Radius of the brightest part of the secondary bow	51.58
Radius of the brightest part of its first spurious bow	53.57

If $3(\sin\phi)^2 = (2+\mu)(2-\mu)$, $\mu\sin\phi' = \sin\phi$, the radius of the geometrical primary bow of the colour corresponding to the index μ will be $4\phi' - 2\phi$; and if $8(\sin\psi)^2 = (3+\mu)(3-\mu)$, $\mu\sin\psi' = \sin\psi$, the radius of the geometrical secondary bow of the colour corresponding to the index μ will be $\pi + 2\psi - 6\psi'$.

According to Fraunhofer (Denkschriften der K. Akademie der Wissenschaften zu München für die Jahre 1814 und 1815. S. 214, 224.) the brightest part of the solar spectrum lies between the lines *D, E*, at a distance of between one-third and one-fourth of *DE* from *D*; and the indices of refraction of water for the lines *D, E*, are 1.33358 and 1.33585 respectively. Therefore, for the brightest part of the solar spectrum the index of refraction of water will be 1.33424. Hence the radii of the geometrical primary and secondary bows will be 41°.53′,9 and 51°.12′,9 respectively.

The theoretical distances of the brightest part of a bow and its first spurious bow from the geometrical bow are as the numbers 1.08 and 3.47. In the primary bow the difference between the radius of the first spurious bow and the radius of the geometrical bow is 1°.27′. Therefore, according to theory, the distance of the primary from the geometrical bow is 27′, or the theoretical radius of the brightest part of the primary is 41°.27′. The observed radius is 41°.32′. Hence the observed place of the primary is 5′ nearer to the geometrical bow than its place as assigned by theory. In like manner the theoretical radius of the brightest part of the secondary bow is found to be 52°.6′. Hence the observed place of the secondary is 8′ nearer to the geometrical bow than its theoretical place.

In a second series of observations, the eyehole of the telescope was covered with a red glass which transmitted light from the least refrangible end of the spectrum nearly up to the line *D*. The points selected for observation were the dark bands and the brightest part of the principal bow. The dark bars could be seen very distinctly, and were easily bisected. Considerable difficulty was, however, still felt in fixing upon the brightest part of a principal bow, on account of its breadth and the want of a symmetrical distribution of light on both sides of the brightest point. An inspection of the results will shew that the latter was subject to considerable uncertainty. All the observations were liable to be affected by a sudden shifting of the bars, which was seen occasionally to take place through a small space to the right or left. The angular distances of as many of the dark bars as could be conveniently observed, are given in order to meet any future calculation of their places. The observations of the points at which the bows appeared to commence, present, as might be expected, great discrepancies; they shew, however, that in every case the illumination extends considerably beyond the place of the geometrical bow. The spectrum seen through the red glass fades much more abruptly from its brightest point towards the more refrangible end than towards the less refrangible end. *N* being the more refrangible end, *M* the brightest point, *L* that at which it begins to fade, *K* the least refrangible end; the indices of those points deduced from the double deviations through a hollow prism having an angle of 66°.22′ are, at *K* 1.3294, at *L* 1.3310, at *M* 1.3322, at *N* 1.3334. The best single equivalent index will probably be about 1.3318.

(A)

Primary bow, seen through red glass, for which it has been assumed that $\mu = 1.3318$, and $\therefore 4\phi' - 2\phi = 42^{\circ}.15'$. Diameter of the cylinder of water = 0.0206 inch.

Limit	42° . 50′	64′	36′	46′	53′	61′	50′	52′	55′	64′
Brightest......	41 . 49	50	51	45	58	54	51	55	45	56
Dark band 1	41 . 8	10	8	2	11	6	4	7	8	7
.... 2	40 . 16	17	19	10	19	18	13	15	...	17
.... 3	39 . 36	37	37	27	37	33	33	32	30	
.... 4	38 . 57	58	58	50	60	58	53			
.... 5	38 . 25	27	26	18	26					
.... 6	37 . 54	50	54	46						
.... 7	37 . 24	22	28							
.... 8	36 . 59	56	58							
.... 9	36 . 31	29	33							
.... 10	36 . 7	3	6							
.... 11	35 . 45	37	43							
.... 12	35 . 21	12								
.... 13	35 . 1									
.... 14	34 . 40									

(B)

Secondary bow, seen through red glass, for which it has been assumed that $\mu = 1.3318$, and $\therefore \pi + 2\psi - 6\psi' = 50^{0}.34'$. Diameter of the cylinder of water = 0.0206 inch.

Limit	49⁰ . 65′	53′	54′	51′	65′	58′	68′
Brightest	51 . 30	27	16	20	30	21	29
Dark band 1	52 . 36	39	36	37	37	40	33
.... 2	54 . 2	10	1	6	8	12	7
.... 3	55 . 19	26	16	25	22	30	25
.... 4	56 . 23	30	26	30			
.... 5	57 . 24	33					
.... 6	58 . 18	32					
.... 7	59 . 10						
.... 8	59 . 56						

In a third series of observations, the Sun's light, after being transmitted through a vertical slit 0.25 inch wide, was received upon a prism distant about 24 feet, having its edge vertical. A second slit also 0.25 inch wide being placed immediately behind the prism, a tolerably pure spectrum was formed. The stream of water was then placed at the distance of about 18 feet from the prism, nearly in the brightest part of the spectrum: and the index of refraction of the rays that fell upon the stream, determined in the same manner as that of the light transmitted through the red glass.

(C)

Primary bow. $\mu = 1.3346$, $\therefore 4\phi' - 2\phi = 41.^{\circ}50',4$. Diameter of the cylinder of water = 0.02105 inch.

Limit	42°. 29′	55′	24′	34′	52′	43′	51′
Brightest	41 . 27	31	26	33	25	25	27
Dark band 1	40 . 49	53	50	52	53	51	52
.... 2	40 . 4	5	3	4	7	3	5
.... 3	39 . 27	28	26	27	28	25	26
.... 4	38 . 51	53	52	54	56	52	...
.... 5	38 . 21	25	22	25	24	20	22
.... 6	37 . 52	55	54	55	57	52	54
.... 7	37 . 25	29	27	29	31	24	
.... 8	37 . 0	4	3	4	5	...	
.... 9	36 . 34	41	38	40	42	36	
.... 10	36 . 11	17	14	17	18	11	
.... 11	35 . 48	55	52	55	57	48	
.... 12	35 . 22	34	32	32	36	28	
.... 13	35 . 4	12	9	11	14	8	
.... 14	34 . 43	53	48	52	55	46	
.... 15	34 . 23	33	27	33	35	26	
.... 16	34 . 3	13	10	13	18	8	
.... 17	33 . 47	55	50	52	58	47	
.... 18	33 . 28	37	33	35	40		
.... 19	33 . 8	19	13	23	22		
.... 20	32 . 53	58	56	58			
.... 21	32 . 36	41	40				
.... 22	32 . 18	25	23				
.... 23	32 . 1	6	5				
.... 24	31 . 34	53	50				
.... 26	31 . 28	32	29				
.... 27	31 . 5	18					
.... 28	30 . 53	65					
.... 29	30 . 38						
.... 30	30 . 24						

(D)

Secondary bow. $\mu = 1{,}33464$, $\therefore\ \pi + 2\psi - 6\psi' = 51^{\circ}.19'$. Diameter of the cylinder of water = 0.02105 inch.

Limit	50°. 13′	7′	30′	57′	19′	8′	47′
Brightest......	51 . 59	59	57	69	50	49	57
Dark band 1	53 . 5	2	6	6	5	5	6
.... 2	54 . 27	23	29	30	28	25	31
.... 3	55 . 36	31	38	39	37	32	36
.... 4	56 . 35	30	38	38	35	35	36
.... 5	57 . 29	25	42	32	30	29	30
.... 6	58 . 19	13	23	22	22	20	21
.... 7	59 . 6	0	11	13	10	8	8
.... 8	59 . 50	45	56	55	55	51	50
.... 9	60 . 33	29	40	38	41		
.... 10	61 . 15	12	22	17	20		
.... 11	61 . 52	50	62	58	59		
.... 12	62 . 32	29	41	36	42		
.... 13	63 . 9	3	17	17	17		
.... 14	63 . 43	39	52	49	51		
.... 15	64 . 18	15	29	25	28		
.... 16	64 . 52	49	62	61			
.... 17	65 . 25	24	37	32			
.... 18	65 . 57	55	71	68			
.... 19	66 . 29	26	45				
.... 20	66 . 63	54	76				
.... 21	67 . 32	26	43				
.... 22	68 . 0						
.... 23	68 . 35						

A fourth series of observations was made with a smaller cylinder of water, the diameter of which is rather uncertain, the tube having been accidentally broken before the observations for determining the diameter of the stream were repeated. At the commencement of the observations it was found that $\mu = 1.33453$; at the conclusion it was found that $\mu = 1.3348$. This shews that either the prism or the stream had been displaced in the interval. The comparison of the observed and theoretical radii has been made with both values of μ. The former of these agrees best with the theory.

(E)

Primary bow. If $\mu = 1.33453$, $4\phi' - 2\phi = 41^{\circ}.52'$. If $\mu = 1.3348$. $4\phi' - 2\phi = 41^{\circ}.49'$. Diameter of the cylinder of water $= 0.0135$ inch.

Limit	42° . $68'$	$67'$	$57'$	$65'$	$77'$	$64'$	$48'$
Brightest	41 . 18	20	18	19	30	17	19
Dark band 1	40 . 34	32	31	33	34	32	34
.... 2	39 . 28	29	31	28	29	29	31
.... 3	38 . 38	39	39	38	40	38	39
.... 4	37 . 52	54	54	54	53	53	53
.... 5	37 . 12	13	14	12	13	12	14
.... 6	36 . 34	34	36	35	35	31	34
.... 7	35 . 56	58	62	60	58	61	61
.... 8	35 . 23	21	28	26	25	29	27
.... 9	34 . 51	51	57	...	53	53	
.... 10	34 . 20	19	28	...	21	22	
.... 11	33 . 49	50	57	...	50	52	
.... 12	33 . 21	20	27	25	24	...	
.... 13	32 . 53	54	58	57	55	55	
.... 14	32 . 26	26	31	30	32		
.... 15	31 . 58	...	62	64			
.... 16	31 . 34	...	39	37			
.... 17	31 . 9	...	11				
.... 18	30 . 44	...	44				
.... 19	30 . 20	...	25				
.... 20	29 . 56	...					
.... 21	29 . 30	...					
.... 22	29	3					

(F)

Secondary bow. If $\mu = 1.33453$, $\pi + 2\psi - 6\psi' = 51^{\circ}.17',5$. If $\mu = 1.3348$. $\pi + 2\psi - 6\psi' = 51^{\circ}.23',3$. Diameter of the cylinder of water $= 0.0135$ inch.

Limit	49°. 53′	35′	63′	13′	48′	48′	58′
Brightest	52 . 26	20	0	6	14	18	26
Dark band 1	53 . 38	36	38	32	40	43	35
.... 2	55 . 35	28	32	30	30	35	29
.... 3	57 . 4	3	1	4	6	9	0
.... 4	58 . 26	23	21	19	25	25	13
.... 5	59 . 38	37	33	32	42	35	37
.... 6	60 . 44	44	32	39	48	43	39
.... 7	61 . 48	48	...	43	55		
.... 8	62 . 54	47	44	44			
.... 9	63 . 48	51					
.... 10	64 . 41	39					
.... 11	65 . 34						
.... 12	66 . 25						

According to theory, the distances of the brightest part of a principal bow and of the 1st and 2nd dark rings from the geometrical bow, are as 1.08; 2.48; 4.4. Whence, knowing the calculated radius of the geometrical bow and the observed radius of the first dark ring, the theoretical radii of the brightest part of the principal bow and that of the second dark ring may be found. In the following comparison of these with the mean of the results obtained by observation, it will be seen that the differences between theory and observation are not greater than might reasonably be expected. It will, however, be remarked that in every instance the observed principal bow is a little nearer to the geometrical bow than theory indicates. This, if not accidental, may be due to an error in pointing, occasioned by the want of symmetry in the distribution of the light in the principal bow.

		Obs. ° ′	Theory. ° ′	° ′
(*A*)	$(4\phi' - 2\phi$		42 . 15)	
	rad. primary	41 . 51,4	41 . 45,4	
	rad. 1st dark ring	41 . 7	———	
	rad. 2nd dark ring	40 . 16	40 . 14,4	
(*B*)	$(\pi + 2\psi - 6\psi'$		50 . 34)	
	rad. secondary ...	51 . 25	51 . 27,5	
	rad. 1st dark ring	52 . 37	———	
	rad. 2nd dark ring	54 . 7	54 . 12	
(*C*)	$(4\phi' - 2\phi$		41 . 50,4)	
	rad. primary	41 . 27,7	41 . 24,7	
	rad. 1st dark ring	40 . 51,4	———	
	rad. 2nd dark ring	40 . 4,4	40 . 5,7	
(*D*)	$(\pi + 2\psi - 6\psi'$		51 . 19,2)	
	rad. secondary ...	51 . 57	52 . 5,3	
	rad. 1st dark ring	53 . 5	———	
	rad. 2nd dark ring	54 . 27,6	54 . 27	
(*E*)	$(4\phi' - 2\phi$	———	41 . 52?	41 . 49?)
	rad. primary	41 . 20	41 . 18	41 . 15
	rad. 1st dark ring	40 . 33	———	———
	rad. 2nd dark ring	39 . 29	39 . 32	39 . 31
(*F*)	$(\pi + 2\psi - 6\psi'$ ...	———	51 . 17,5?	51 . 23,3?)
	rad. secondary ...	52 . 16	52 . 18,5	52 . 21
	rad. 1st dark ring	53 . 37,4	———	———
	rad. 2nd dark ring	55 . 31,3	55 . 26	55 . 21

W. H. MILLER.

St. John's College,
Dec. 14, 1840.

XVI. *On the Foundation of Algebra*, No. II. *By* AUGUSTUS DE MORGAN, F.R.A.S., F.C.P.S.; *of Trinity College; Professor of Mathematics in University College, London.*

[Read *November* 29, 1841.]

IN presenting to the Society a continuation of the Paper on the Foundation of Algebra, printed in Vol. VII, p. 173, I wish to make the principal point of the new communication, which is the filling up of an unfinished difficulty of the old one, subservient to such a view of the transition from semi-logical to logical algebra as may perhaps be useful to any one who may hereafter have to deal with an unexplained result. By the semi-logical algebra I mean the ordinary science, in which the explanations are insufficient to include $\sqrt{-1}$; and in which therefore the results, though always intelligible when $\sqrt{-1}$ disappears, can only be considered as demonstrated upon the assumption that the symbolical laws of algebra must in some, though an unknown, manner, admit of a wider explanation.

The first step to logical algebra is the separation of the rules of the ordinary science from its principles, or rather of its laws of operation from the explanation of the symbols operated upon or with. As far as I can see (and I believe no writer has professed to throw together in one place every thing that is essential to algebraical process) the laws of operation are as follow:

1. The literal symbols, *a*, *b*, *c*, &c. have no necessary relation except this, that whatever any one of them may mean in any one part of a process, it means the same in every other part of the same process.

2. The sign = is the only one of which the explanation is requisite in the art of operation: it signifies an assertion of identity of operative effect, and gives the right to substitute one side for the other, when desired. Its use implies a postulate, the only one demanded: that $a = b$ gives $A = B$ whenever A is derived from a by the same operations in the same order, which produce B from b.

3. The signs + and − are opposite in effect; what one does the other undoes: and 0 is the symbol of a pair of such opposite operations having been performed. Thus $+a - a = 0$. And such operations are convertible in their order: thus $+a-b+c = +c-b+a = -b+c+a$, &c.

4. The signs × and ÷ (or any substitutes for them) are opposite in effect: and 1 is the symbol of a pair of such opposite operations having been performed. Thus $\times a \div a = 1$. And these operations are also convertible in their order: thus

$$\times a \div b \times c = \times c \div b \times a = \div b \times c \times a, \text{ \&c.}$$

5. The operations × and ÷ are of a *distributive* character, when performed upon the results of the operations + and −. Thus

$$(+a) \times (+b-c) = (+a) \times (+b) + (+a) \times (-c), \text{ \&c.}$$

6. Like signs (+ and −) produce + in all cases, and unlike signs −. And like signs (× and ÷) produce × in all cases, and unlike signs ÷. And each pair of signs is, relatively to its own set, distributive.

7. The signs 0 and 1 may themselves be considered as subjects of operation, and $1+1$ is abbreviated into 2, $1+1+1$ into 3, $1+1+1+1$ into 4, and so on.

8. The laws by which the symbol a^b is used are $a^b \times a^c = a^{b+c}$ and $(a^b)^c = a^{bc}$.

I believe the preceding rules to be neither insufficient nor redundant, though I should be noways surprised to see them proved both the one and the other; least of all if it were the latter.

The most remarkable point in this separation is that the laws of operation prescribe much less of connexion between the successive symbols $a+b$, ab, and a^b, than a person who has deduced these laws from an arithmetical explanation would at first think sufficient. The only connexion between the two fundamental operations of $+$ and $\times$ is contained in $a(b \pm c) = ab \pm ac$, and though from this it is a demonstrable identity that

$$a + a + a + \overset{\text{abbreviated.}}{\ldots\ldots} = (1 + 1 + 1 + \ldots) \times a,$$

which establishes a connexion between $+$ and $\times$ when one of the factors is derived solely from 1, yet it leaves the general symbol ab, when neither a nor b is so derived, apparently more free of the meaning of $a + b$ than any one would predict it ultimately must be: while a^b is still less connected with its predecessor ab. I shall now examine the manner in which this independence of the three operations has acted in the explanations which have appeared.

Choosing a unit-line of arbitrary length and direction, and signifying by A or (a, α), a line of a units in length inclined to the unit-line at an angle α, it is well known that an explanation can be given, under which the preceding laws of operation become real consequences of real conceptions. And it is worth stopping to note that the art of operation, previously to the explanation of its symbols, is precisely what Dugald Stewart imagined every mathematical science to be, namely, a pure consequence of definitions, which upon other definitions might have been another thing. This opinion was not, and perhaps is not, without its followers: but I think it will hardly, in any mind, stand the test of a comparison of any one mathematical science with the purely technical algebra, which is rigorously founded upon definitions. By itself, this method of operation, this algebra of rules without meaning, is no more of a science than the use of the well-known toy called the Chinese puzzle, in which a prescribed number of forms are given, and a large number of different arrangements, of which the outlines only are drawn, are to be produced. Perhaps a dissected map or picture would be a still better illustration: a person who puts one of these together by the backs of the pieces, and therefore is guided only by their forms, and not by their meanings, may be compared to one who makes the transformations of algebra by the defined laws of operation only: while one who looks at the fronts, and converts his general knowledge of the countries painted on them into one of a more particular kind by help of the forms of the pieces, more resembles the investigator and the mathematician.

Mr. Warren, in his explanation*, and Dr. Peacock, in his interpretation*, of the algebraical symbols, have both obtained the symbols $a+b$ and ab independently of each other as to their meaning: while both, to obtain the meaning of the symbol a^b, have had recourse to the fundamental derivation from a, aa, aaa, &c. The consequence is, that while both establish most completely the ordinary forms of algebra, neither is prepared to consider a^b where b is other than what answers to the positive or negative number or fraction of the semi-logical algebra. Mr. Warren, who carefully avoids all *interpreted* results, and whose work is as complete a succession of consequences from explanations adequate to the results as that of any professedly arithmetical algebraist, has therefore totally avoided the use of such a symbol as $\epsilon^{\theta\sqrt{-1}}$, using instead $\left(\frac{1}{1}\right)^{\frac{\theta}{2\pi}}$, a new convention, derived from the roots of unity. Dr. Peacock, making use virtually of the equation

$$\cos\theta + \sqrt{-1}\sin\theta = (\cos 1 + \sqrt{-1}\sin 1)^\theta,$$

and denoting $\cos 1 + \sqrt{-1}\sin 1$ by ϵ, is able fully to interpret all results arising from $\epsilon^\theta = \cos\theta + \sqrt{-1}\sin\theta$, and to prove this equation as a consequence of the laws of operation. Both writers would consider $\epsilon^{\theta\sqrt{-1}} = \cos\theta + \sqrt{-1}\sin\theta$, as an equation resembling $-(-A) = A$ in ordinary algebra, of which the first side, known to be the same symbol as the second, can only receive its explanation from the second. And we see that the complete independence of the explanations or interpretations of $a+b$ and ab leads (in the works alluded to) to a full and satisfactory account of their properties, while the derivation of a^b from ab ends in a partial and insufficient notion of the meaning of the symbol itself. There is something disappointing in the first-mentioned circumstance, since the mind naturally looks, in the most extensive view of the subject, for the prototypes of those analogies and modes of derivation which were of such essential use in the more bounded science: but at the

* I use these words in the same sense as in my last Paper.

same time the power of adhering to the modes of derivation of the partial view is too dearly paid for by a want of generality in the general one.

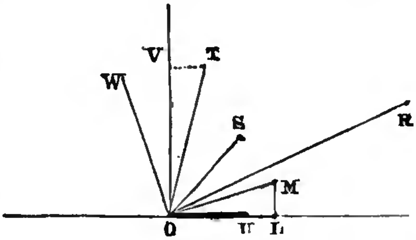
In my last Paper I pointed out that the analogy of the definitions of $a + b$ and ab in arithmetic and algebra was perfect, insomuch that, by an abstraction of the subject-matter of the former from those definitions, the remaining words make definitions which will equally apply to both views of the science. In fact, $a + b$ is in both, a direction to do with a what must be done with 0 to make b; while ab is a direction to do with a what must be done* with 1 to make b. I now proceed to disengage a^b from its partial dependence on ab, and having established an independent definition, to examine the analogies which exist between a^b in the ancient and modern view of the subject.

Let $R = (r, \rho)$, be a line of r units inclined to the unit-line at the angle ρ; and this being $r \cos \rho + r \sin \rho \sqrt{-1}$, let $r \cos \rho = R_x$, $r \sin \rho = R_y$. It is in our power to suppose this line given by means of another, $R' = (r', \rho')$, by the conditions $R_x' = \phi(r, \rho)$, $R_y' = \psi(r, \rho)$, ϕ and ψ being known functions, from which r and ρ can be determined in terms of r' and ρ'. The second line may be called the determinant of the first, and the first line may be said to be determined from the second. Now supposing only the operation + to have been defined in its most general sense, we have from every form of ϕ and ψ the means of instituting a new process, as follows. Instead of adding two lines, add their determinants, and let the sum of the determinants be the determinant of a new line. If (r, ρ), (s, σ), be the given lines, and (t, τ) the determined line, we have then

$$\phi(t, \tau) = \phi(r, \rho) + \phi(s, \sigma), \quad \psi(t, \tau) = \psi(r, \rho) + \psi(s, \sigma).$$

* The most analogical view of a^b is not a natural one, owing to the idea of a logarithm being made subsequent to that of an exponent. But if the notion of a logarithm were obtained, prior to the definitions of algebra, from two continuous linear motions, which severally give equal increments in equal times, the one in difference and the other in ratio, the exponent obtained from them would first enter as a logarithm, and would always retain the character of a logarithm rendered into numbers, just as $\sin^{-1} A$ always retains that of an angle rendered into numbers. Hence, $a^{\log b}$ or $b^{\log a}$, which are the same things, would be defined from ϵ, introduced in the first process, as follows: $a^{\log b}$ is what arises from doing to a that which must be done with ϵ to form b.

If we look for that system in which the newly determined line is the product, RS, or *$(r, \rho) \times (s, \sigma)$ or $(rs, \rho + \sigma)$, we find that we must have $\phi(r, \rho) = \log r$, $\psi(r, \rho) = \rho$. The system of logarithms is meant to be the purely arithmetical one, and for reasons of numerical convenience, the base ϵ is taken, and the angle is measured by the ratio of the arc to the radius. This species of determinant is what should be called the *logarithm* of (r, ρ); but, considering it desirable to retain the word *logarithm* for purposes purely numerical, I should prefer to call $\left(\sqrt{(\log r)^2 + \rho^2},\ \tan^{-1}\frac{\rho}{\log r}\right)$ the *logometer* of (r, ρ). All this would throw no light upon the general meaning of the sign $\times$, but it leads immediately to that comprehensive definition of R^S or $(r, \rho)^{(s, \sigma)}$, which Mr. Warren might have adopted, to the introduction of $\epsilon^{\theta\sqrt{-1}}$, without creating the smallest flaw in his well-secured title to be considered as having most strictly adhered to explained definitions only: and which Dr. Peacock might have regarded as the complete interpretation of every symbolical result in which an exponent occurs that cannot be laid down on one side or the other of the unit-line. That definition is as follows: R^S means the line of which the logometer is obtained by multiplying together S and the logometer of R. Thus, OU being the unit-line, let it be required to lay down OR^{OS}. Let OL be the logarithm of OR, and ML the arc of $\angle ROU$ (rad. OU): then OM is the logometer of OR. Take OT a fourth proportional to OU, OM, OS, and $\angle TOU = \angle SOU + \angle MOU$; then TO is the logometer of the result required. Place a line of which the logarithm is TV at an angle whose arc is OV, and that line, OW, is the one represented by OR^{OS}. The fundamental laws of operation are so readily established that I do not feel it necessary to enter upon them; and the equation



$\epsilon^{\theta\sqrt{-1}} = \cos\theta + \sqrt{-1}\sin\theta$ is a mere corollary of the definition; for the logometer of ϵ, or $(\epsilon, 0)$ is $(1, 0)$, and $(1, 0) \times \theta\sqrt{-1}$ or $(1, 0) \times \left(\theta, \frac{\pi}{2}\right)$

* It is to be understood that all operations upon the small letters are those of common arithmetic.

is $\left(\theta, \frac{\pi}{2}\right)$, which is the logometer of a line whose length has $\theta \cos \frac{\pi}{2}$, or 0, for its logarithm, inclined at the angle $\theta \sin \frac{\pi}{2}$, or θ. Hence $\epsilon^{\theta\sqrt{-1}}$ is a unit of length, inclined at an angle θ; or $\cos\theta + \sqrt{-1}\sin\theta$.

It will appear rather against the preceding definition, that it points out $\epsilon^{\theta\sqrt{-1}}$ to signify the same as $\cos\theta + \sqrt{-1}\sin\theta$, whatever ϵ and π may be: for there is nothing in the preceding demonstration which has reference to any particular value of these constants. And, in reality, as far as this one definition is concerned, and its consequences, there would be no limitation upon the meanings of π and ϵ. But when—having invented this indefinite mode of constructing $(1, \theta)$, which leads us to our result whatever may be ϵ or π, and in fact contains a direct and inverse use of ϵ which prevents the value of that letter from affecting the result—we equate this mode of producing $(1, \theta)$ to a definite mode derived from another definition, we must expect to see the indefinite character of the former changed, by the introduction of new conditions, into the definite character of the latter. But this would be no answer to the difficulty: for it would be admitting a new fundamental rule among the laws of operation. Let the reasonableness of the expectation just alluded to be ground for an assumption, and we see that π (two right angles) and ϵ are connected by the equation $\epsilon^{\sqrt{-1}} = \cos 1 + \sqrt{-1}\sin 1$, or ϵ depends upon the angle denoted by 1, which is all that is necessary. But I say that this equation is a consequence of the whole set of definitions only, without any new assumption. In the first place, it is easily shewn that $R^s = R \times R \times R \ldots (s \text{ times})$ when $S = (s, 0)$ and s is integer. Next, from the definition of multiplication it immediately follows that

$$\cos s\theta + \sqrt{-1}\sin s\theta = (\cos\theta + \sqrt{-1}\sin\theta)(\cos\theta + \sqrt{-1}\sin\theta)\ldots(s \text{ times});$$

$$\text{whence } \cos s\theta + \sqrt{-1}\sin s\theta = (\cos\theta + \sqrt{-1}\sin\theta)^s;$$

whence it is $\cos 1 + \sqrt{-1}\sin 1$, *and that only*, which, raised to the integer power of s, gives $\cos s + \sqrt{-1}\sin s$; for it may readily be shewn that no other line (r, ρ) can have the same s^{th} power as $(1, 1)$. It is then the result of the definitions of addition and multiplication that nothing but $\cos 1 + \sqrt{-1}\sin 1$, raised to the power of s (integer), gives

$(1, s)$: it is the consequence of the definition of an exponential operation, *considered apart from the rest,* that $\epsilon^{\sqrt{-1}}$ raised to the power of s, gives $(1, s)$: the whole system therefore requires that $\epsilon^{\sqrt{-1}} = \cos 1 + \sqrt{-1} \,.\, \sin 1$; which is thus proved previous to the equation of $\epsilon^{\theta\sqrt{-1}}$, and $\cos\theta + \sqrt{-1}\sin\theta$ from the definition of an exponent. Hence ϵ depends only upon the angular unit, which may be a degree, a minute, a right angle, or any other, provided that ϵ be taken accordingly. The proof that $\epsilon = 1 + 1 + \frac{1}{2} + \ldots$, when the angle 1 is that which has an arc equal to the radius, must be a subsequent matter.

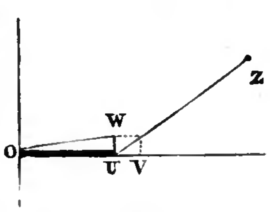
If $\lambda(r, \rho)$ represent the logometer, or complete algebraical logarithm, of (r, ρ), the equation $(r, \rho) = \epsilon^{\lambda(r, \rho)}$ is an identical one; for the logometer of ϵ, or $\lambda(\epsilon, 0)$, being $(1, 0)$, say that (t, τ) is that of (r, ρ), whence, by definition, $(1, 0) \times (t, \tau)$ or (t, τ) is the logometer also of $\epsilon^{\lambda(r, \rho)}$. Hence, making $\theta = \pi$, we have $\epsilon^{\pi\sqrt{-1}} = -1$, and taking the obvious truth that lines equal (both in length and direction) have the same logometers, we have

$$\pi = \frac{\lambda(-1)}{\sqrt{-1}},$$

a proposition which, not many years since, was one of the mysteries of analysis. It is now a very simple geometrical proposition: the first side means a line of π units laid down positively on the unit-line; the second side means the logometer of a negative unit turned back through a right angle. Now the logometer of a negative unit is a line of π units erected positively perpendicular to the unit-line: whence the identity of the two sides is manifest.

On the analogy of the complete definition of R^s with that of a^b in arithmetic, it can only be said that, so far as the latter is intelligible, it is seen to coincide with the former: while the former itself introduces an element which seems, up to this time, to defy analogy drawn from arithmetic; namely, the representation of a projection on the unit-line by a *logarithm*, and of one on the perpendicular to it by an *angle*. We see how this happens in the deduction of $\epsilon^{\theta\sqrt{-1}} = \cos\theta + \sqrt{-1}\sin\theta$, and we also see that the general definition of an exponent may be derived from the idea of *exposition* (to use an old phrase) of one symbol by another, in such a manner as to reduce multiplication to the result of addition of exponents. The combination of a line, not an exponent, with one which is an exponent, by the operation newly learnt, or which might have been learnt, from combining two exponents by the known operation $+$, is then obviously natural, and its result completes the definitions of algebra in their most comprehensive form. But it is satisfactory to find that the matter is not thus exhausted; and it remains a subject of speculation how it arises that a line *perpendicular* to the unit-line has the same relation to an *angle* which one *on* the unit-line has to the *logarithm of a length*. The following considerations tend, as far as they go, to give some idea of the origin of this circumstance.

Let us go on to the generation of quantities by infinite numbers of infinitely small elements, premising that nothing will be said which may not easily be altered into the language of limits by those who object to the infinitesimal phraseology. Every line (r, ρ) can only be formed by addition of equal* elements in one way, namely, that expressed by $\int_0^r (dr, \rho)$, ρ being constant. Let us now consider what takes place when (r, ρ) becomes $(r+dr, \rho+d\rho)$. This is obviously equivalent to multiplying the first line by $(1+\frac{dr}{r}, d\rho)$, or successively by $(1+\frac{dr}{r}, 0)$ and $(1, d\rho)$. The first multiplication alters length only, in the proportion of $r+dr$ to r; and answers to multiplying by OV, UV being $dr : r$. Now, OU being the unit-line, make $UW = d\rho$ in linear units; whence, by the conventions of angular measurement, OW is $(1, d\rho)$, neglecting the differential of the second order by which OW differs from OU. These *ratiunculæ* (such was the term applied to differentials so employed when the theory of logarithms was first explained), UV and UW, being each used in the multipliers n times in succession, we have resolved the contemporaneous transitions (linear and angular) into distinct and (if we please) successive transitions. If we begin with OU, or $(1, 0)$, and proceed through n multiplications, and make $dr : r$ always the same, and $=\mu$, we have



* Remember that 'equal' means 'same in length and direction.'

$$(\{1+\mu\}^n, nd\rho) = (1+\mu)^n . (1+d\rho\sqrt{-1})^n \ldots\ldots (A).$$

Now let n be infinite, and let $(1+\mu)^n = r$, $nd\rho = \rho$, from which we find

$$n = \frac{\log r}{\mu} = \frac{\rho}{d\rho}, \quad (1+d\rho\sqrt{-1})^n = \epsilon^{\rho\sqrt{-1}},$$

all following from the assigned laws of operation. Hence $r\epsilon^{\rho\sqrt{-1}}$ is the representative of a line r inclined at an angle ρ; while $\log r$ and ρ, each in its own way, and to a different radix, may be considered as a register of the number of transitions by which we pass from $(1, 0)$ to (r, ρ). The term *logarithm* itself, as is well known, is a consequence of a similar notion of comparison of numbers by the registration of the 'numbers of the ratios' by which we pass from unity to those numbers. The ratiunculæ μ and $d\rho$ must be in the proportion of $\log r$ and ρ, and the line formed by adding WU and UV, or $\mu + d\rho\sqrt{-1}$, is one which gives these two ratiunculæ for its two projections. This line repeated n times gives $\log r + \theta\sqrt{-1}$, the logometer of (r, θ) as I have called it. Should any objection be taken to that term, perhaps the words compound logarithm might be preferable. Observe, that this derivation of the logometer is independent of the second side of (A), and might be introduced previously to the expression of (r, θ) in the form $r\epsilon^{\theta\sqrt{-1}}$.

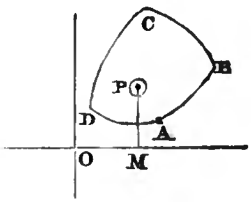
I have throughout avoided considering the ambiguous values of symbols, a thing for which there is no necessity, as has been frequently shewn, and as I noted in my last Paper. The more I think on this subject, the better satisfied do I feel, that the new algebra should have no symbols of double or multiple value whatsoever; that is, that the meaning of each elementary symbol should not be considered as complete, unless it expresses the amount of revolution from the unit line by which it is to be made to attain its direction, as well as that direction itself. Undoubtedly, after a time, the student should be shewn how to drop this part of the definition; but this he will better be able to do than to take it up after a previous training, which has never introduced it. This is an important point to those who believe as I do, that it will not be long before the new algebra is introduced into elementary instruction; and it is the more important, because there are some new species of ambiguities altogether peculiar to the most general view, and which must remain such until further inquiry points out the mode of dealing with them. These last can hardly receive due attention, unless they are carefully distinguished from the previous and well-known cases of the same kind; which will be only done by adopting that system of definitions which destroys the latter altogether.

ADDITION.

A THEOREM of M. Cauchy, which is well known to the readers of Liouville's Journal, by the comparatively easy demonstration which MM. Sturm and Liouville have there given of it, may be set in so clear a point of view by the complete algebra, that I here add a demonstration of it. This theorem belongs essentially to the complete system of algebra, as will be evident from its enunciation.

As before, $Z = (z, \zeta)$ or $z\epsilon^{\zeta\sqrt{-1}}$, means that Z is a line of the length z inclined at an angle ζ to the unit-line.

THEOREM. Let ϕZ be any function of Z, and let $Z = x + y\sqrt{-1}$ give $\phi Z = p + q\sqrt{-1}$. Also, within the whole of the figure *ABCD*, its contour included, let ϕZ, $\phi' Z$, &c. never become infinite, x and y being the co-ordinates of any point within it. Let any point be called a radical point which makes ϕZ or $\phi(x + y\sqrt{-1}) = 0$. In carrying a point in the positive direction of revolution round the contour *ABCD*, let the fraction $\frac{p}{q}$ change sign *by passing through zero* k times from + to −, and l times from − to +: but let it never change sign by passing through $\frac{0}{0}$, that is, let there be no radical point on the contour itself, and neglect altogether the cases in which it changes sign by passing through ∞. Then the number of radical points contained *within* the contour is $\frac{1}{2}(k - l)$.



Encircle any point P by an infinitely small contour, on which let a point be carried round P. Four cases arise; neither p nor q vanishes within or on this contour; p vanishes but not q; q vanishes but not p: or both vanish.

If neither p nor q vanish, there is never change of sign in either (for by hypothesis they do not become infinite), and the theorem is true for this infinitely small contour: for k and l are both $= 0$, and there is no radical point.

If p alone vanish, the curve $p = 0$ (p being a function of x and y,) passes through the small contour at a single or multiple point: and $\frac{p}{q}$ may change sign at those points of the contour through which the curve passes; but the fraction always becomes 0 and never ∞. There are then as many changes of sign from + to − as from − to +, and the theorem is true: for $k = l$, and there is no radical point.

If q alone vanish, the curve $q = 0$ passes through the point: and every thing is as in the last, except that $\frac{p}{q}$ always becomes ∞ when it changes sign. Hence the theorem is true; for k and l are each $= 0$, and there is no radical point.

Lastly, let there be a radical point within, but not on, the contour: which it is evident may be supposed to contain only one radical point. Let $x = \mu$, $y = \nu$ at the radical point, and let Z be the radius vector drawn from the origin to a point in the contour, and R that drawn from the radical point to the same point of the contour. If then

$$\mu + \nu\sqrt{-1} = A = (a, \alpha);$$

we have, using the extended system of algebra,

$$Z = A + R, \text{ or } (z, \zeta) = (a, \alpha) + (r, \rho), \text{ or } z\,\epsilon^{\zeta\sqrt{-1}} = a\,\epsilon^{\alpha\sqrt{-1}} + r\,\epsilon^{\rho\sqrt{-1}},$$

r being infinitely small. Now let $\phi(A + R)$, or ϕZ, be capable of being expanded into the series

$$B_0 R^m + B_1 R^{m+1} + B_2 R^{m+2} + \ldots\ldots$$

in which, on account of the value of R, we need only consider the first term $B_0 R^m$.

By our hypothesis m is an integer, and there are m radical points in one, answering to m equal roots of $\phi Z = 0$. Also, B_0 being (b_0, β_0), we have

$$B_0 R^m = b_0 r^m \{\cos(m\rho + \beta_0) + \sin(m\rho + \beta_0) . \sqrt{-1}\}.$$

whence $\frac{p}{q} = \cot(m\rho + \beta_0)$.

Now while ρ goes through a whole revolution, $m\rho + \beta_0$ passes from β_0 to $2m\pi + \beta_0$ through m complete revolutions, and changes sign $2m$ times from + to - passing through 0 each time: but it never changes from − to + except by passing through ∞. Hence $k = 2m$, $l = 0$, $\frac{1}{2}(k - l) = m$, which verifies the theorem, since there are m roots within the contour.

Next, let the whole figure $ABCD$ be divided into an infinite number of infinitely small figures, with no other limitation than that no radical point is to fall upon one of the lines of division: and let a point move round each of the infinitely small figures in the positive direction of revolution. It is clear that the expression $\frac{1}{2}(\Sigma k - \Sigma l)$ will not be altered if we remove all the internal division lines and leave only the external contour $ABCD$: for each internal line is described by two points moving in opposite directions, and wherever one point adds a unit to Σk, the other adds one to Σl. Hence the value of $\Sigma k - \Sigma l$ can be found by finding that of $k - l$ for the external contour only.

If we suppose ϕZ to be rational and integral, say $= A_0 Z^n + A_1 Z^{n-1} + \ldots$, and if we make the contour in question a circle with the origin as a center, and a radius so great that the highest term $A_0 Z^n$ need be the only one retained, we find that $\frac{p}{q} = \cot(n\zeta + a_0)$, which gives, as before, in a revolution, $k = 2n$, $l = 0$; whence the whole number of roots of $\phi Z = 0$ is neither more nor less than n.

It is easily deduced from the preceding that the number of real roots of $\phi x = 0$ lying between $x = a$ (the less) and $x = b$ the greater, is the number of *vanishing* changes of sign from − to + which take place while x passes from a to b in the quotient of

$$\phi x - \phi'' x \frac{y^2}{2} + \phi^{\text{iv}} x \frac{y^4}{2.3.4} - \ldots \text{ divided by } \phi' x . - \phi''' x \frac{y^2}{2.3} + \phi^{\text{v}} x \frac{y^4}{2.3.4.5} - \ldots$$

y being an infinitely small positive constant: provided that neither a nor b is a root. This result, however, would be of little general use: in fact, this theorem of Cauchy requires an examination of the contour which is equivalent in that which must be made of the axis to find the real roots. It makes the examination of a line equivalent to that of the whole included space; but does not profess to help in that examination. But in the important case in which the contour is a rectangle with sides parallel to the axes, or when it is desired to find all the roots of the form $x + y\sqrt{-1}$ in which x lies between given limits, and y between other given limits, this theorem is a complete reduction of the question to that of finding the number of real roots of four equations which lie between given limits, one pair for each equation. It thus supplies the theoretical desideratum which Fourier and Sturm have left.

A. DE MORGAN.

UNIVERSITY COLLEGE, LONDON,
October 12, 1841.

XVII. *An Enquiry into the Causes which led to the Fatal Accident on the Brighton Railway* (Oct. 2, 1841), *in which is developed A Principle of Motion of the greatest importance in guarding against the Disastrous Effects of Collision under whatever circumstances it may occur.* *By the Rev.* J. POWER, M.A., *Fellow and Tutor of Trinity Hall, Cambridge.*

[Read *November* 29, 1841.]

WHEN accidents have occurred on railways, in the majority of instances some cause has been immediately apparent, to which the occurrence might be reasonably imputed; but the fatal accident which took place on the Brighton Railway, in October last, the cause of which still lies buried in the greatest obscurity, forms a remarkable exception to the above rule.

In meditating on the possible causes of this accident, I have arrived at some Dynamical Results, which will be given in the course of this investigation, and which are, as I conceive, of the greatest importance to the public safety, inasmuch as, by attention to them on the part of Engineers, the disastrous consequences of collision may be very materially diminished.

In order to pursue the proposed inquiry with effect, it was necessary in the first place to ascertain, as accurately as circumstances would permit, the true *history* of the accident. With this view, I have examined with great care the report of the Coroner's Inquest as given in the Times newspaper, and have succeeded, to my own satisfaction, in forming a tolerably consistent narrative out of the disjointed materials of which the evidence is composed.

It would be tedious to occupy the attention of the reader with the critical details of the above examination, and I shall therefore proceed at once to lay before him the historical conclusions at which I have arrived, so far as they are connected with the object I have in view.

I will take up the narrative at that point where the train, with two engines in front, was making its way from the Horley Station in the direction of Brighton, and was proceeding with apparent safety at the rate of about 30 miles an hour. The leading, or *Pilot* engine as it is termed, was a small four-wheeled engine which had been put on at the Horley Station, in order to afford a temporary assistance to the large six-wheeled engine which had brought the train from London. I shall call these the first and second engine respectively, as they occurred in the order of the train, though it may be worth mentioning, that, in the evidence of Goldsmith, the driver of the pilot engine, these terms are used in the inverted sense, the large original engine being named the first, and the smaller engine, which was subsequently prefixed, being named the second.

On approaching a certain cutting, called "the Copyhold Cutting," and at a distance from it of about half a mile, the driver of the second or large engine turned off his steam, his motive for so doing being, that he might reserve his steam for the latter part of the way to Brighton, when he should be deprived of the assistance of the other engine. By this operation the speed was reduced from 30 to about 20 or 25 miles an hour, and the train was continuing to proceed without any inconvenience over the half mile which intervened between the place where the steam of the second engine was turned off and the entrance of the cutting.

Before entering the cutting, a labourer on the road was observed to hold up his hand, the usual sign for slackening speed. Upon this, the driver of the first engine turned off *his* steam "to within half an inch," an interval which (as the regulator in closing the aperture, through which the steam enters into the cylinder, moves over a quandrantal arc of seven inches radius[1],) corresponds to a deficiency of about $\frac{1}{20}$ from the whole extent of the range.

* See description of Stephenson's Locomotive Engine in the new edition of Tredgold on the Steam Engine.

Immediately after this, the driver of the first engine perceived a different motion in his engine, which, to use his own expression, "wavered backwards and forwards," and in a very short interval of time, the fore wheels were thrown off the rail, causing the engine to upset, and giving rise to the accident, with all its frightful concomitants. The motion which preceded the upset of the engine, is described by the driver of the second engine as a *rocking* motion, and by a labourer who viewed it from the road, as a *jumping up and down of the fore wheels.* The short period of time for which it lasted was estimated by the driver of the second engine, and the person who viewed it from the road, at half a minute; and though the driver of the first engine stated that it "lasted only an instant," some allowance must here be made for the vagueness of language, and we shall probably be not far from the truth, if we regard it as an interval of very small but sensible duration. It appeared also, by the evidence of one of the engine drivers, that at the time the accident happened, "the pilot engine was doing very little work, *merely keeping tight the chain;*" from which incidental expression it may be inferred with certainty, that the two engines were connected by a chain, which was at liberty to be loose or tight according to the distance between the two engines*, a fact, be it observed, which is of the greatest importance in the explanation which is about to be offered.

It further appeared in evidence, that the engine had been examined and found to be in perfect order; and though the driver of the second engine gave it as his opinion, that the accident was occasioned by clay, or some greasy substance, lying on the rail, yet it appears by the evidence of the person whose duty it was to inspect this portion of road, that the rail was perfectly clean and had nothing slippery upon it, that it was, moreover, on a bed of sand, and not on clay. The preceding rains had indeed rendered it a prudent precaution that the train should come steadily over this portion of road, and such was alleged to be the meaning of the signal which the man made by holding up his hand, a practice which

* I have lately seen a chain of this description connecting two engines drawing a heavy train on the Birmingham railway, allowing a considerable variation of distance (to the extent perhaps of about a foot) between the engines.

had been adopted for some days previous. It appears also that the rate at which the train was proceeding was nothing more than the usual rate, which was well warranted by the sound state of the rail. Indeed, no danger, or cause of danger, appears to have been suspected by any person belonging to the establishment up to the moment the accident commenced.

The first engine was, indeed, described as top-heavy, having been recently filled with water, but this was nothing more than must usually occur under similar circumstances, that is to say, when a pilot engine is first attached.

We have, then, before us a train of carriages (including the large six-wheeled engine, whose steam had previously been turned off), drawn at a comparatively moderate rate by a four-wheeled engine in front, and in the apparent absence of predisposing causes, we have to account for the continued jumping motion, which was observed to take place in the front engine upon the driver's turning off his steam.

Now it occurred to me, that the motion described was exactly such as would have resulted from a series of jolts or moderate impulses communicated at the back of the engine, provided they were communicated at a point considerably lower than its centre of gravity; and I propose to show, in the first place, how an impulse from behind communicated lower than the centre of gravity would cause the front engine to lift up its fore wheels; and secondly, to point out how, under the circumstances of the case, a *succession* of such ascents of the fore wheels might possibly have taken place. For, be it observed, the phenomenon to be explained is not so much the final overthrow of the engine, as the preceding jumping motion, or series of jumps, which led to it; a single jump might, no doubt, have been sufficient to cause the accident, but the probability of danger arising from a single jump, would be incomparably smaller than that which would arise from any cause which rendered possible a continued series of them, such as was in fact observed to take place on the present occasion.

I shall demonstrate hereafter the following important proposition, namely, that there exists at the back of a locomotive carriage of any kind, a point, at which if a horizontal impulse be communicated, no jumping motion of the kind we are considering can possibly result, but if the impulse be communicated, at a point lower than this, the carriage will lift up its fore wheels, and if higher than this, it will lift up its hind wheels.

Professor Willis has been so kind as to fit up, at my request, a small model, which affords a most satisfactory experimental proof of the truth of this proposition*, which I previously arrived at by mathematical considerations.

If we neglect the mass of the pair of wheels and axle about which the rotation is supposed to take place, a simple mathematical calculation suffices to show, that the point of quiescence, or that at which the horizontal impulse must be applied, in order that no rotatory motion of the kind we are considering may be impressed, will be situated in the same horizontal level with the centre of gravity of the carriage. But since in locomotive engines the wheels and axles are of considerable mass, (in Stephenson's locomotive the driving pair of wheels and the axle connecting them weigh about a ton and a quarter,) it was desirable to ascertain how far this circumstance would cause the point of quiescence to deviate from the level of the centre of gravity.

The problem then becomes more complicated, but the result of the investigation shows, that when the rail is regarded as perfectly smooth, the deviation of the point of quiescence from the centre of gravity of the whole carriage, including the wheels and axle, is nothing, whatever be the mass of the latter. The roughness of the rail may however cause it to deviate slightly below this level, but even in the case of perfect roughness, the deviation is so slight that in practice it may be safely neglected.

The mathematical details connected with this part of the subject will be given in the sequel; and I now proceed to mention a *second* principle, which is essential to the solution I am about to offer, and which, in the absence of sufficient data, I assume rather as a very probable hypothesis, than as a mathematically demonstrated theorem like the former. The principle I assume is this, that if the two

* The experiment was exhibited at the time the communication was read.

engines, both having their steam turned off, were started with the same velocity, the lighter would be retarded more rapidly than the heavier.

If indeed we assume that the machinery is similar in both, the principle will be readily allowed; for the rapidity of the retardation will be as the retarding causes directly, and as the mass of the engine inversely.

The principal retarding causes when the steam is turned off are, 1st, the resistance of the air; 2dly, the friction of the parts of the machinery sliding over one another, the principal of which is, probably, the friction of the pistons within the cylinders; 3dly, the roughness of the rail, that is to say, such retarding causes as may act in the manner of minute obstacles lying upon it; 4thly, the friction on the axes.

Now, though the effect of the two last causes will be nearly as the masses of the engines, the two former may be regarded as pretty nearly the same, the machinery of the engines being supposed similar in the two cases; on the whole, therefore, the retarding causes in the smaller engine will be diminished in a less ratio than its mass; it will consequently be retarded more rapidly than the other.

Let us now apply the principles which have been laid down to the case before us.

It will be recollected that the engines were connected by a chain, which admitted of being tight or loose as the distance between them was greater or less: it will be recollected also, that the pilot engine was described as "top-heavy", which makes it highly probable that the frame, to which the chain is usually applied at the middle, and the buffers, (or disks which are brought in contact when the engines approach each other), at the two sides, was considerably lower than the centre of gravity of the engine. Lastly, it will be recollected, that the steam of the large engine had been previously turned off. Let us now consider what will take place when the steam of the small engine in front is turned off.

The small engine being retarded more rapidly than the large one following it, the engines will be brought nearer and nearer to each other, the connecting chain in the mean time becoming slacker and slacker, and by the time the buffers are brought in contact, a finite difference of velocities will be generated: this will occasion the *first jolt*, which being applied at the back of the pilot engine considerably lower than its centre of gravity, will cause it to lift up its fore wheels. By the elasticity of the buffers, (which, when attached to engines, are not usually furnished with springs like those attached to the carriages), the velocity of approach will be immediately converted into a velocity of separation, according to the laws of elasticity and impact, and, for the moment, the front engine will be again driven ahead of the other; but by the continued excess of the retardation of the first engine, the velocity of separation will at length be reconverted into a velocity of approach, giving rise to a *second jolt*, and occasioning the front engine a second time to lift up its fore wheels; and the same process might be repeated a great number of times in succession.

In the above explanation, for the sake of simplicity, I have omitted the consideration of the effect of the train following the second engine; but since the carriages are exempt from one great cause of retardation to which the engines are subject, namely, the friction of internal machinery, it is clear that they would, if left to themselves, be retarded less rapidly than the engines, whence it is easy to see, that the effect of the train of carriages will tend to push on the second engine, and increase the effect which has been described.

I have supposed, moreover, that the engine has time to resume its natural position after each jolt before the succeeding jolt is communicated, but if the jolts succeed one another more rapidly than the natural time of an ascent and descent of the front wheels, it is easy to see how, under favourable circumstances, they might conjoin their effects so as to increase the angle of elevation very considerably. The most favourable case for producing this effect, would be, when the second jolt is communicated at the precise moment when the front wheels have attained their highest elevation due to the motion impressed by the first, and so on for the third and following jolts.

These successive ascents of the fore wheels were exactly the phenomena which it was our object to account for, being such as were observed to precede the overthrow of the engine, and such as no doubt will be attended with the greatest danger whenever they occur. Indeed it is manifest, that if during the ascent of the fore wheel, any accidental cause should give the engine at the same time a hitch in a horizontal direction, the fore wheels must be thrown off the rail, as in fact took place in the present instance, occasioning the overthrow of the engine and the disastrous consequences which ensued.

Whether the above be the true solution of the accident it is possible that a variety of opinions may exist; but I conceive, no one will doubt the importance of the principles which have been developed in the course of this investigation, and which give rise to the following practical conclusions:—

1. In the construction both of the engines and carriages care should be taken that the centre of gravity of each engine or carriage be about as low as the horizontal frame to which the buffers and links are attached.

2. If conformity with the above rule be attended with practical inconvenience, the same object might be attained by placing the buffers at the proper height, by means of strong additional frame-work, connected with and rising from the general horizontal frame.

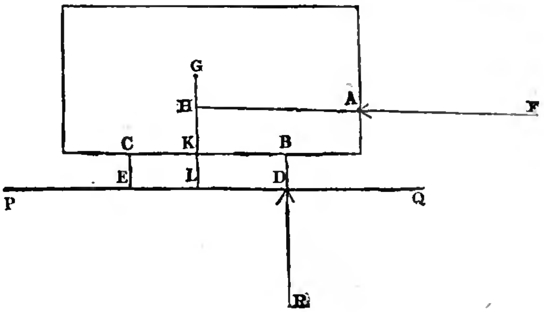
I am not sure that a single pair of opposing buffers placed mid-way between the rails, would not be better than two pairs of buffers placed one immediately over each rail, in order to avoid any tendency to rotatory motion in a horizontal plane, which any inequality of action in the latter might occasion. But there may be practical objections to this arrangement with which I am unacquainted.

3. A further means of diminishing the danger would be to shorten as much as possible the connecting chains, in order to constrain the engines and carriages to move with the same velocity, and to prevent the accumulation of any finite difference of velocities between any two engines or carriages throughout the train*.

* I have lately observed on the Birmingham railway that the buffers of the contiguous *carriages* are forced into immediate contact, and the connecting chains made as tight as possible by means of a screw-power. Why might not the same mode of connection be adopted when a pilot engine is attached, instead of a loose chain, which appears to be the usual practice? If injury to the machinery be feared, arising from the jarring vibrations which would accompany this contact, these might be prevented by furnishing the buffers of the engines with springs similar to those of the carriages.

I will now give the mathematical details to which allusion has been made.

It will be as well to begin with the very simple case of a body whose centre of gravity is G, supported on a perfectly smooth horizontal plane PQ by two props BD, CE, and urged by a horizontal impulsive force F at the point A. Draw $GHKL$ vertical.



Let $GH = h$,

$DL = a$,

$GL = b$,

M the mass,

k its radius of gyration about G,

V the horizontal velocity communicated to D,

α the angular velocity about D resulting from the impact.

Since the velocities communicated to G are

$$a\alpha \text{ vertically upwards, and}$$

$$V - b\alpha \text{ horizontally forwards;}$$

we have by the usual principles of motion,

$$M(V - b\alpha) = F,$$

$$Ma\alpha = R,$$

$$Mk^2\alpha = Fh - Ra,$$

R denoting the vertical reaction at D.

Hence $M(a^2 + k^2)\alpha = Fh$,

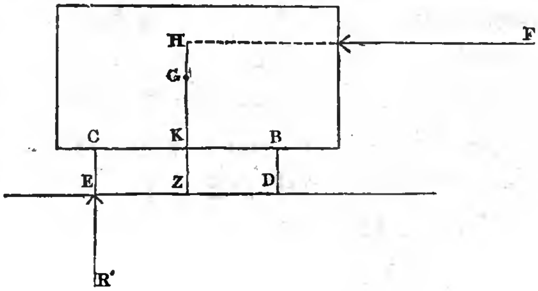
$$\alpha = \frac{Fh}{M(a^2 + k^2)},$$

$$V = \frac{F}{M} + b\alpha = \frac{F}{M} \cdot \frac{a^2 + k^2 + bh}{a^2 + k^2}.$$

Consequently, if h is nothing, no angular motion will result from the shock; this is the case when the horizontal impulse F is directed through the centre of gravity.

But if the direction of F passes lower than the centre of gravity, an angular velocity about D must necessarily result, the magnitude of which is proportionate to the distance h at which the impulse passes below G.

If the impulse passes *above* G, h changing its sign, α assumes a negative value, which is impossible so long as D is supposed to remain in contact with the plane; we must in this case regard E as the point which remains in contact with the plane, and calling R' the vertical reaction at E,



a' the distance EL,

α' the angular velocity about E,

we have, as before,

$$Ma'\alpha' = R',$$

$$Mk^2\alpha' = Fh - R'a';$$

whence $M(a'^2 + k^2)\alpha' = Fh$,

$$\alpha' = \frac{Fh}{M(a'^2 + k^2)},$$

which shows, as before, that if the direction of F passes through G, no angular rotation will be communicated; and further, that if the direction of F passes higher than G, an angular velocity, whose magnitude is proportionate to the distance GH, will be generated, by virtue of which D will be carried upwards, E remaining in contact with the plane.

The preceding results may be regarded as near approximations to the truth, in the case of a wheeled carriage, when the mass of the wheels is inconsiderable compared with the mass of the carriage.

But as the wheels and connecting axles in locomotive engines are very massive, it may be useful to inquire what influence the mass of the wheels and axle, about which the rotatory motion takes place, may have in modifying the preceding results.

The carriage with its wheels not constituting a single rigid body, as in the last case, the problem becomes much more complicated, and it is extremely difficult to avoid sources of error in applying to it the same formulæ for the motion of rigid bodies, as were used in the preceding example. On this account I prefer treating it by the method of Lagrange.

Let us suppose the whole mass of the engine to be projected upon a vertical plane parallel to the direction of the rails, and that a horizontal impulse F from behind causes it to lift up its front wheels and rotate about its hind or driving pair of wheels.

Let x, y be the horizontal and vertical co-ordinates of any point m of the carriage, deprived of its hind pair of wheels and their connecting axle, referred to a fixed origin behind the carriage.

x', y' those of any point m' of the hind wheels and axle;

u, v the horizontal and vertical velocities communicated to m by the impact;

u', v' the same for m'.

By D'Alembert's Principle, the momenta subject to the conditions of equilibrium are

$$F, \; -mu, \; -\&c. \; -m'u', \; -\&c. \text{ horizontal;}$$
$$\text{and } \; -mv, \; -\&c. \; -m'v', \; -\&c. \text{ vertical.}$$

Hence, naming $\bar{x}$ the horizontal co-ordinate of the point of application of F, we have by the principle of virtual velocities,

$$\left.\begin{array}{l} F\delta\bar{x} - \Sigma(mu\delta x) - \Sigma(m'u'\delta x') \\ \quad - \Sigma(mv\delta y) - \Sigma(m'v'\delta y') \end{array}\right\} = 0. \;\ldots\ldots (1).$$

Let V be the linear velocity communicated to the axis of the hind wheels.

α the angular velocity communicated to the carriage about the axis of the hind wheels, tending to diminish x and increase y.

α' the angular velocity communicated to the hind wheels about their axis, tending to increase x' and diminish y'.

$\delta\theta$, $\delta\theta'$ any small virtual angles of rotation of the carriage and hind wheels in direction of the angular velocities α, α' respectively.

δs the corresponding horizontal space described by the axle of the hind wheels.

Then if the rail be perfectly smooth the variations $\delta\theta$, $\delta\theta'$, δs are independent of each other. But if the rail be perfectly rough, so that the wheel remains in perfect rolling contact with it during the shock, $\delta\theta'$ and δs are connected by the equation $\delta s = r\delta\theta'$, r being the radius of the wheel.

Again, assuming the moveable projection of the axis as a new origin, let

ξ, η be the co-ordinates of m;

ξ', η' be the co-ordinates of m'.

We have, manifestly,

$$\begin{aligned} u &= V - \eta a, \\ v &= \xi a, \\ u' &= V + \eta' a', \\ v' &= -\xi' a'. \end{aligned}$$

Also,

$$\begin{aligned} \delta x &= \delta s - \eta\delta\theta, \\ \delta y &= \xi\delta\theta, \\ \delta x' &= \delta s + \eta'\delta\theta', \\ \delta y' &= -\xi'\delta\theta'. \end{aligned}$$

Again, if a, b be the horizontal and vertical co-ordinates of the centre of gravity of the carriage exclusive of the hind wheels, and h the vertical distance below this centre at which F passes, we have

$$\delta\bar{x} = \delta s - (b - h)\,\delta\theta.$$

Substituting these values, the equation (1) becomes

$$\left.\begin{aligned} 0 = F\{\delta s - (b-h)\delta\theta\} &- \Sigma\{m(V - \eta a)(\delta s - \eta\delta\theta)\} \\ &- \Sigma\{m\xi^2 a\delta\theta\} \\ &- \Sigma\{m'(V + \eta' a')(\delta s + \eta'\delta\theta')\} \\ &- \Sigma\{m'\xi'^2 a'\delta\theta'\} \end{aligned}\right\} \ldots\ldots (2).$$

Hence if the rail be perfectly smooth, equating to zero the coefficients of the arbitrary variations δs, $\delta\theta$, $\delta\theta'$, we obtain

$$\left.\begin{aligned}0 &= F - V\Sigma m + \alpha\Sigma m\eta - V\Sigma m' - \alpha'\Sigma m'\eta',\\ 0 &= -F.(b-h) + V\Sigma m\eta - \alpha\Sigma\{m(\xi^2+\eta^2)\},\\ 0 &= -V\Sigma m'\eta' - \alpha'\Sigma m'(\eta'^2+\xi'^2).\end{aligned}\right\}\ldots\ldots(3).$$

If $M = \Sigma m =$ mass of the carriage without the hind wheels and axle;
$M' = \Sigma m' =$ mass of the hind wheels and axle;
k the radius of gyration of M about its own centre of gravity;
k' that of M' about its axle.

Since $\Sigma m\eta = Mb$,

$$\Sigma m'\eta' = 0,$$

$$\Sigma m(\xi^2+\eta^2) = M(a^2+b^2+k^2),$$

$$\Sigma m'(\xi'^2+\eta'^2) = M'k'^2,$$

the equations (3) above reduce themselves to

$$0 = F - (M+M')V + Mb\alpha,$$

$$0 = -F(b-h) + MbV - M(a^2+b^2+k^2)\alpha,$$

$$0 = -\alpha' M' k'^2.$$

The last gives $\alpha' = 0$, which shows, as we might have expected, that the rail, being perfectly smooth, has no power of impressing a finite angular velocity by a horizontal reaction.

The two former give by the elimination of V,

$$\alpha = \frac{F}{M}\cdot\frac{h - \dfrac{M'b}{M+M'}}{a^2+k^2+\dfrac{M'b^2}{M+M'}}.$$

If the rail be perfectly rough, the equation (2) becomes by the above reductions,

$$0 = \{F - (M+M')V + Mb\alpha\}\,\delta s$$
$$+ \{Fh - Fb + MbV - M(a^2+b^2+k^2)\alpha\}\,\delta\theta$$
$$- \alpha' M' k'^2\,\delta\theta',$$

with which must be combined the conditions,

$$\delta\theta' = \frac{\delta s}{r},$$

$$\alpha' = \frac{V}{r}.$$

Substituting these values of $\delta\theta'$ and α', and equating to zero the coefficients of δs, $\delta\theta$, we obtain

$$0 = F - (M + M')\,V + Mb\alpha - M'V.\frac{k'^2}{r^2},$$

$$0 = Fh - Fb + MbV - M(a^2 + b^2 + k^2)\,\alpha.$$

The former of which gives

$$V = \frac{F + Mb\alpha}{M + M'\left(1 + \dfrac{k'^2}{r^2}\right)}.$$

Substituting this in the second, and putting

$$q = \frac{M'\left(1 + \dfrac{k'^2}{r^2}\right)b}{M + M'\left(1 + \dfrac{k'^2}{r^2}\right)}, \text{ we find}$$

$$\alpha = \frac{F}{M}\cdot\frac{h - q}{a^2 + k^2 + qb}.$$

Hence in general,

$$\alpha = \frac{F}{M}\cdot\frac{h - q}{a^2 + k^2 + qb},$$

where $q = 0$ when $\dfrac{M'}{M} = 0$,

$= \dfrac{M'b}{M + M'}$ when the rail is smooth,

$= \dfrac{M'\left(1 + \dfrac{k'^2}{r^2}\right)b}{M + M'\left(1 + \dfrac{k'^2}{r^2}\right)}$ when the rail is perfectly rough.

Since $\frac{M'b}{M+M'}$ is the vertical distance of the centre of gravity of $M + M'$ below that of M; it follows that in the second case, as in the first, the impulse must pass through the centre of gravity of the *whole carriage* $M + M'$, in order that no rotatory motion may be impressed.

In the third case, if, for greater practical convenience, we wish to determine the distance ($q_{,}$ suppose) of the point of quiescence below the centre of gravity of the *whole carriage*, denoting by $b_{,}$ the height of the centre of gravity of the *whole carriage* above the level of the axle, we have

$$b_{,} = b - \frac{M'b}{M+M'} = \frac{Mb}{M+M'}$$

$$q_{,} = q - \frac{M'b}{M+M'} = \frac{M'\left(1+\frac{k'^2}{r^2}\right)b}{M+M'\left(1+\frac{k'^2}{r^2}\right)} - \frac{M'b}{M+M'}$$

Reducing and substituting for b its value in $b_{,}$, we find

$$q_{,} = \frac{M' . \frac{k'^2}{r^2} . b_{,}}{M + M' + M'\frac{k'^2}{r^2}} = \frac{M'}{M+M'} . \frac{k'^2}{r^2} b_{,} \text{ very nearly.}$$

If, with Tredgold, we suppose $\left.\begin{array}{r} M' = 1\frac{1}{4} \\ \text{and } M + M' = 12 \end{array}\right\}$ tons,

and at a rough estimate take $\frac{k'}{r} = \frac{1}{2}$, we find $q_{,} = .006 b_{,}$ very nearly.

Thus, if $b_{,} = 12$ inches, $q_{,} = .072$ inches;

if $b_{,} = 18$ inches, $q_{,} = .108$;

if $b_{,} = 2$ feet, $q_{,} = .144$.

Hence we see, that the weight of the wheels and axles, and the roughness or smoothness of the rail, make no difference perceptible in practice; and that in order to ensure the absence of rotatory motion in a vertical plane, arising from a horizontal collision, it is necessary and sufficient that the centre of the buffers should be placed as nearly as may be on the same level with the centre of gravity of the engine or carriage.

If we suppose the rotatory motion impressed to be very small, it is easy to calculate approximately the ascent of the elevated wheel.

Neglecting $\frac{M'}{M}$ and calling θ the small angle of elevation at the time t, we shall have approximately,

$$Mk^2 \cdot \frac{d^2\theta}{dt^2} = -Pa,$$

$$\frac{d^2 \cdot (a\theta)}{dt^2} = -g + \frac{P}{M},$$

P denoting the vertical pressure at the rail.

Hence
$$\frac{d^2\theta}{dt^2} = -\frac{ag}{a^2+k^2},$$

$$\frac{d\theta}{dt} = \text{const.} - \frac{agt}{a^2+k^2}$$

$$= \frac{Fh - Magt}{M(a^2+k^2)},$$

the initial angular velocity being $\frac{Fh}{M(a^2+k^2)}$.

Integrating again, so that θ is 0 when $t = 0$, we obtain

$$\theta = \frac{2Fht - Magt^2}{2M(a^2+k^2)}.$$

Hence, θ attains its maximum value when $Fh - Magt = 0$, or $t = \frac{Fh}{Mag}$, and at that moment the value of θ is $\frac{F^2h^2}{2M^2ag(a^2+k^2)}$.

If we suppose the centre of gravity to be situated half way between the two axles, multiplying by $2a$, we find for the extreme elevation of the wheels, when the disturbance is small, the expression

$$\frac{F^2h^2}{M^2g(a^2+k^2)};$$

a quantity, which it is desirable to render as small as possible, in order to ensure safety to the engine or carriage under ordinary circumstances.

It has lately been the subject of discussion, whether, by increasing the distance between the bearings, the proportionate increase of the linear ascent of the wheel, due to a given angular rotation, would not increase the danger of running off the rail.

The preceding result shows, that no such danger is to be feared, but that, on the contrary, the increased linear ascent of the wheel, due to the greater length of revolving radius, is far more than compensated by the diminution of the angular velocity itself; and, as regards the comparative safety of four-wheeled and six-wheeled engines, it shows a decided advantage in favour of the latter:—

1st, because they admit more readily of the diminution of h by placing the centre of gravity lower.

2nd, on account of their greater mass.

3rd, on account of the greater distance between their centre of gravity and the axles of the fore and hind wheels.

4th, on account of the increased value of k, the radius of gyration.

P. S. Since the above communication was read, an equally distressing event has taken place on the Great Western Railway, which affords a striking illustration of the importance of these principles. The persons who travelled upon or next to the luggage trucks were the unfortunate victims on this occasion, and the fatal consequences appear to have arisen from the *rolling of the trucks one over another*, on the train being unexpectedly stopped by a fall of earth lying upon the railway. Had the buffers been placed at the proper horizontal level, this rolling motion could not have taken place, and the loss of human life might have been prevented.

J. POWER.

TRINITY HALL,
April 12, 1842.

XVIII. *Discussion of the Question:—Are Cause and Effect successive or simultaneous? By the Rev.* W. WHEWELL, B.D., *Master of Trinity College, and Professor of Moral Philosophy.*

[Read *March* 14, 1842.]

I HAVE at various times laid before this Society dissertations on the metaphysical grounds and elements of our knowledge, and especially on the foundations of the science of mechanics. As these speculations have not failed to excite some attention, both here and elsewhere, I am tempted to bring forward in the same manner some additional disquisitions of the same kind. Indeed, the immediate occasion of the present memoir is of itself an evidence that such subjects are not supposed to be without their interest for the general reader; for I am led to the views and reasonings which I am now about to lay before the Society, by some remarks in one of our most popular Reviews, (*The Quarterly Review*, Article on the History and Philosophy of the Inductive Sciences. June 1841.) A writer of singular acuteness and comprehensiveness of view has there made remarks upon the doctrines which I had delivered in the "Philosophy of the Inductive Sciences," which remarks appear to me in the highest degree instructive and philosophical. I am not, however, going here to discuss fully the doctrines contained in this critique. With respect to its general tendency, I will only observe, that the author does not accept, in the form in which I had given it, the account of the origin and ground of necessary and universal truths. I had stated that our knowledge is derived from Sensations and Ideas; and that Ideas, which are the conditions of perception, such as *space, time, likeness, cause*, make universal and necessary knowledge possible; whereas, if knowledge were derived from Sensation alone, it could not have those characters. I have moreover enumerated a long series of Fundamental Ideas as the bases of a corresponding series of sciences, of which sciences I have shown also, by an historical survey, that they claim to possess universal truths, and have their claims allowed. I have gone further: for I have stated the Axioms which flow from these Fundamental Ideas, and which are the logical grounds of necessity and universality in the truths of each science, when the science is presented in the form of a demonstrated system. The Reviewer does not assent to this doctrine, nor to the argument by which it is supported; namely, that Experience cannot lead to universal truths, except by means of a universal Idea supplied by the mind, and infused into the particular facts which observation ministers. He considers that the existence of universal truths in our knowledge may be explained otherwise. He holds that it is a sufficient account of the matter to say that we pass from special experience to universal truth in virtue of "the inductive propensity—the irresistible impulse of the mind to generalize *ad infinitum*." I shall not here dwell upon very strong reasons which may be assigned, as I conceive, for not accepting this as a full and satisfactory explanation of the difficulty. Instead of doing so, I shall here content myself with remarking, that even if we adopt the Reviewer's expressions, we must still contend that there are *different forms* of the *impulse of the mind to generalize*, corresponding to each of the Fundamental Ideas of our system. These Fundamental Ideas, if they be nothing else, must at least be accepted as a classification of the modes of action of the Inductive Propensity,—as so many different paths and tendencies of the Generalizing Impulse: and the Axioms which I have stated as the express results of the Fundamental Ideas, and as the steps by which those Ideas make universal truths possible, are still no less worthy of notice, if they are stated as the results of our Generalizing Impulse; and as the steps by which that Impulse, in its many various forms, makes universal truths possible. The Generalizing Impulse in that operation by which it leads us to the Axioms of geometry, and to those of mechanics, takes very different courses; and these courses may well deserve to be separately studied. And perhaps, even if we accept this view of the philosophy of our knowledge, no simpler or clearer way can be found of describing and distinguishing these fundamentally different operations of the Inductive Propensity, than by saying, that in the one case it proceeds according to the Idea of Space, in another according to the Idea of Mechanical Cause; and the like phraseology may be employed for all the other cases.

This then being understood, my present object is to consider some very remarkable, and, as appears to me, novel views of the Idea of Cause which the Reviewer propounds. And these may be best brought under our discussion by considering them as an attempt to solve the question, Whether, according to our fundamental apprehensions of the relation of Cause and Effect, effect follows cause in the order of time, or is simultaneous with it.

At first sight, this question may seem to be completely decided by our fundamental convictions respecting cause and effect, and by the axioms which have been propounded by almost all writers, and have obtained universal currency among reasoners on this subject. That the cause must precede the effect,—that the effect must follow the cause,—are, it might seem, self-evident truths, assumed and assented to by all persons in all reasonings in which those notions occur. Such a doctrine is commonly asserted in general terms, and seems to be verified in all the applications of the idea of cause. A heavy body produces motion by its weight; the motion produced is subsequent in time to the pressure which the weight exerts. In a machine, bodies push or strike each other, and so produce a series of motions; each motion, in this case, is the result of the motions and configurations which have preceded it. The whole series of such motions employs time; and this time is filled up and measured by the series of causes and effects, the effects being, in their turn, causes of other effects. This is the common mode of apprehending the universal course of events, in which the chain of causation, and the progress of time, are contemplated as each the necessary condition and accompaniment of the other.

But this, the Critic remarks, is not true in *direct* causation. "If the antecedence and consequence in question be understood as the interposition of an interval of time, however small, between the action of the cause and the production of the effect, we regard it as inadmissible. In the production of motion by force, for instance, though the effect be cumulative with continued exertion of the cause, yet each elementary or individual action is, to our apprehension, *instanter* accompanied with its corresponding increment of momentum in the body moved. In all dynamical reasonings no one has ever thought of interposing an instant of time between the action and its resulting momentum; nor does it appear necessary." This is so evident, that it appears strange it should have the air of novelty; yet, so far as I am aware, the matter has never before been put in the same point of view. But this being the case, the question occurs, how it is that time *seems* to be employed in the progress from cause to effect? How is it that the opinion of the effect being subsequent to the cause has generally obtained? And to this the Critic's answer is obvious:—it is so in cases of indirect or of *cumulative* effect. If a ball *A* strikes another, *B*, and puts it in motion, and *B* strikes *C*, and puts it in motion, *A*'s impact may be considered as the cause, though not the direct cause, of *C*'s motion. Now time, namely the time of *B*'s motion after it is struck by *A*, and before it strikes *C*, intervenes between *A*'s impact and the beginning of *C*'s motion: that is, between the cause and its effect. In this sense, the effect is subsequent to the cause. Again, if a body be put in motion by a series of impulses acting at finite intervals of time, all in the same direction, the motion at the end of all these intervals is the effect of all the impulses, and exists after they have all acted. It is the accumulated effect, and subsequent to each separate action of the cause. But in this case, each impulse produces its effect instantaneously, and the time is employed, not in the transition from any cause to its effect, but in the intervals between the action of the several causes, during which intervals the body goes on with the velocity already communicated to it. In each impulse, force produces motion: and the motion goes on till a new change takes place, by the same kind of action. The force may be said, in the language employed by the Critic, to be transformed into momentum; and in the successive impulses, successive portions of force are thus transformed; while in the intervening intervals, the force thus transformed into momentum is carried by the body from one place to another, where a new change awaits it. "The cause is absorbed and transformed into effect, and therein treasured up." Hence, as the Writer says, "The time lost in cases of indirect physical causation is that consumed in the movements which take place among the parts of the mechanism set in action, by which the active forces so transformed into mechanism are transported over intervals of space to new points of action, the motion of matter in such cases being regarded as a mere carrier of force":—and when force is directly counteracted by force, their mutual destruction must be conceived, as the Reviewer says, to be instantaneous. We can therefore hardly resist his conclusion, that men have been misled in assuming sequence as a feature in the relation of cause and effect; and we may readily assent to his suggestion, that sequence, when observed, is to be held as a sure indication of indirect action, accompanied with a movement of parts.

But yet if we turn for a moment to other kinds of causation, we seem to be compelled at every step to recognize the truth of the usual maxim upon this subject, that effects are subsequent to causes. Is not poison, taken at a certain moment, the cause of disorder and death which follow at a *subsequent* period? Is not a man's early prudence often the cause of his prosperity in *later* life, and his folly, though for a moment it may produce gratification, *finally* the cause of his ruin? And even in the case of mechanism, if, in a clock which goes rightly, we alter the length of the pendulum, is not this alteration the cause of an alteration which *afterwards* takes place in the rate of the clock's going? Are not all these, and innumerable other cases, instances in which the usual notion of the effect following the cause is verified? and are they not irreconcileable with the new doctrine of cause and effect being simultaneous?

In order to disentangle this apparent confusion, let us first consider the case last mentioned, of a clock, in which some alteration is made which affects the rate of going.

So long as the parts of the clock remain unaltered, its rate will remain unaltered; and any part which is considered as capable of alteration, may be considered as, if we please, the cause of the unaltered rate, by being itself unaltered. But we do not usually introduce the positive idea of cause, to correspond with this negation of change. If we speak of the rate as unaltered, we may also say that it is so because there is *no cause* of alteration. The steady rate is the indication of the absence of any cause of alteration; and the rate of going measures the progress of time, in a state of things in which causes of change are thus excluded. If an alteration takes place in any part of the clock, once for all, the rate is altered; but the new rate is steady as the old rate was, and, like it, measures the uniform progress of time. But the difference between the new rate and the old is occasioned by the difference of the parts of the clock; and the new rate may very properly be said to be caused by the change of the parts, and to be subsequent to it: for it does prevail after the change, and does not prevail before.

But how is this view to be reconciled with the one just quoted from the Reviewer, and, as it appeared, satisfactorily proved by him; according to which all mechanical effects are simultaneous with their causes, and not subsequent to them? We have here the two views in close contact, and in seeming opposition.

In the going of a clock, the parts are in motion; and these motions are determined by forces arising from the form and connexion of the parts of the mechanism. Each of the forces thus exerted at any instant produces its effect at the same instant; and thus, so far as the term *cause* refers to such instantaneous forces, the cause and the effect are simultaneous. But if such instantaneous forces act at successive intervals of time, the motion during each interval is unaltered, and by its uniform progress measures the progress of time. And thus the motion of the machine consists of a series of intervals, during each of which the motion is uniform, and measures the time; separated from each other by a series of changes, at each of which the change measures the instantaneous force, and is simultaneous with it. And if, in this case, we suppose, at any point of time, the instantaneous forces to cease, the succession of them being terminated, from that point of time the motion would be uniform. And since the rate of the motion in each interval of time is determined by the instantaneous force which last acted and by the preceding motion, the rate of the motion in each interval of time is determined by all the preceding instantaneous forces. Hence, when the series of instantaneous forces stops, the rate at which the motion goes on permanently, from that point of time, is determined by the antecedent series of such forces, which series may be considered as an aggregate cause; and hence it appears, that the *permanent* effect is determined by the *aggregate* cause; and in this sense the effect is subsequent to the cause.

Thus we obtain, in this case, a solution of the difficulty which is placed before us. The instantaneous effect or change is simultaneous with the instantaneous force or cause by which it is produced. But if we consider a series of such instantaneous forces as a single aggregate cause, and the final condition as a permanent effect of this cause, the effect is subsequent to the cause. In this case, the cause is immediately succeeded by the effect. The cause acts in time: the effect goes on in time. The times occupied by the cause and by the effect succeed each other, the one ending at the point of time at which the other begins. But the time which the cause occupies is really composed of a series of instants of uniform motion interposed between instantaneous forces; and during the time that this series of causes is going on, to make up the aggregate cause, a series of effects is going on to make up the final effect. There is a progressive cause and a progressive effect which go on together, and occupy the same finite time; and this simultaneous progression is composed of all the simultaneous instantaneous steps of cause and effect. The aggregate cause is the sum of the progression of causes; the final effect is the last term of the progression of effects. At each step, as the Reviewer says, cause is transformed into effect; and it is treasured up in the results during the intermediate intervals; and the time occupied is not the time which intervenes between cause and effect at each step, but the time which intervenes between these transformations.

I have supposed forces to act at distinct instants, and to cease to act in the intervals between; and then, the aggregate of such intervals to make up a finite time, during which an aggregate force acts. But if the action of the force be rigorously continuous, it will easily be seen that all the consequences as to cause and effect will be the same; the discontinuous action being merely the usual artifice by which, in mathematical reasonings, we obtain results respecting continuous changes. It will still be true, that the uniform motion which takes place after a continuous force has acted, is the effect subsequent to the cause; while the change which takes place at any instant by the action of the force, is the instantaneous effect simultaneous with the cause.

It may be objected, that this solution does not appear immediately to apply: for the motion of a clock is not uniform during any portion of the time. The parts move by intervals of varied motion and of rest; or by oscillations backwards and forwards; and the succession of forces which acts during any oscillation, or any cycle of motion, is repeated during the succeeding oscillation or cycle, and so on indefinitely; and if an alteration be made in the parts, it is not a change once for all, but recurs in its operation in every cycle of the motion.

But it will be found that this circumstance does not prevent the same explanation from being still applicable with a slight modification. Instead of uniform motion in the intervals of causation, we shall have to speak of *steady going:* and instead of considering all the forces which affect the motion as causes of change of uniform motion, we shall have to speak of changes in the parts of the mechanism as causes of *change of rate of going*. With this modification, it will still be true, that any instantaneous cause produces its instantaneous effect simultaneously, while the permanent effect is subsequent to the change which is its cause. The steady going of the clock is assumed as a normal condition, in which it measures the progress of time; and in this assumption, the notion of cause and effect is not brought into view. But a steady rate thus denoting the mean passage of time, a change in the rate indicates a cause of change. The *change of rate*, as an instantaneous *transition* from one rate to another, is *simultaneous* with the change in the parts. But then the *changed rate* as a continued *condition* in which, no new change supervening, the rate again measures the progress of time, is *subsequent* to the change of parts, for it begins when that ends, and continues when the progress of that has ceased.

If, however, this be a satisfactory solution of the difficulty in the case of mechanism, how shall we apply the same views to the other cases? Growth, the effect of food, is subsequent to the act of taking food; disorder, the effect of poison, is subsequent to the introduction of poison into the system. Can we say that the animal would continue unchanged if it were not to take food; and that food is the cause of a change, namely, of growth? This is manifestly false; for if the animal were not to take food, it would soon perish. But the analogy of the former case, of the clock, will enable us to avoid this perplexity. As we assumed a steady rate of going in the clock to be the measure of time when we considered the effect of mechanism, so we assume a steady rate of action in the animal functions to be the measure of the progress of time when we consider the causes which act upon the development and health of animals. Digestion, and of course nutrition, are a part of this normal condition; they are involved in the steady going of the animal mechanism, and we must suppose these functions to go regularly on, in order that the animal may preserve its character of animal. Food and digestion may be considered as causes of the continued existence of the animal, in the same way in which the form of the parts of a clock is the cause of the steady going of a clock. And when we come to consider causes of change, this kind of causation, which produces a normal condition of things, merely measuring the flow of time, is left out of our account. We can conceive an uniform condition of animal existence, the animal neither growing nor wasting. This being taken as the normal condition, any deviation from this condition indicates a cause, and is taken as the evidence and measure of the cause of change. And thus, in a growing animal, the food partly keeps the animal in continued animal existence, and partly, and in addition to this, causes its growth. Food, in the former view, is always circulating in the system, and is supposed to be uniformly administered; the cycles of nutrition being merged in the notion of uniform existence, as the oscillations of the pendulum in a clock are merged in the notion of uniform going; and the elementary steps of nutrition which are, in this view, supposed to take place at each instant, produce their instantaneous effect, for they are requisite in the cycle of animal processes which goes on from instant to instant. But on the other hand, in considering growth, we compare the state of an animal with a preceding state, and consider the nutriment taken in the intervening time as the cause of the change: hence this nutriment, as an aggregate, is considered as the cause of growth of the animal; and in this view the effect is subsequent to the cause. But yet here, as in the case of mechanism, the progressive effect is simultaneous, step by step, with the progressive cause. There is a series of operations; as for instance, intussusception, digestion, assimilation, growth: each of these is a progressive operation; and in the progress of each operation, the steps of the effect and the instantaneous forces are simultaneous. But the end of one operation is the beginning of the next, or at least in part, and hence we have time occupied by the succession. The end of intussusception is the beginning of digestion, the end of digestion the beginning of assimilation, and so on. These aggregate effects succeed each other; and hence growth is subsequent to the taking of food; though each instantaneous force of animal life, no less than of mechanism, produces an effect simultaneous with its action. Each of these separate operations is an aggregate operation, and occupies time; and each aggregate effect is a condition of the action of the cause in the next operation.

Again; if an animal in a permanent condition, neither waxing nor wasting, may be taken as the normal state in which the functions of life measure time, in order that we may consider growth as an effect, to be referred to food as cause; we may, for other purposes, consider, as the normal condition, an animal waxing and then wasting, according to the usual law of animal life: and we must take this, the healthy progress of an animal, as our normal condition, if we have to consider causes which produce disease. If we have to refer the morbid condition of an animal to the influence of poison, for example, we must consider how far the condition deviates from what it would have been if the poison had not been taken into the frame. The usual progress of the animal functions including its growth, is the measure of *time;* the deviation from this usual progress is the indication of *cause;* and the effect of the poison is subsequent to the cause, because the poison acts through the cycle of the animal functions just mentioned, which occupies time; and because the taking the poison into the system, not any subsequent action of the animal forces in the system, is considered as the event which we must contemplate as a cause. To resume the analogy of the clock: the rate of the clock is altered by altering the parts; but this alteration itself may occupy time; as if we alter the rate of a clock by applying a drop of acid, which gradually eats off a part of the pendulum, the corrosion, as an aggregate effect, occupies time; and the rates before and after the change are separated by this time. But the application of the drop is the cause; and thus, in this case the final effect is subsequent to the cause, though here, as in the case of mechanism, the instantaneous forces always produce a simultaneous effect.

Thus we have in every case a *uniform* state, or a state which is considered as uniform, or at least *normal;* and which is taken as the indication and measure of *time;* and we have also *change*, which is contemplated as a deviation from uniformity, and is taken as the indication and measure of *cause*. The uniform state may be one which never exists, being purely imaginary; as the case in which no forces act; and the case in which animal functions go on permanently, the animal neither growing nor wasting. The normal state may also be a state in which change is constantly taking place, as, in fact, even a state of motion is a state of change; such states also are, in a further sense, that of a clock going by starts, and that of an animal constantly growing: in these cases the changes are all merged in a wider view of uniformity, so that these are taken as the normal states. And in all these cases, successive changes which take place are separated by intervals of time, measured by the normal progress; and each change is produced by some *simultaneous* instantaneous cause. But taking the cause in a larger sense, we group these instantaneous causes, and perhaps omit in our contemplation some of the intervening intervals; and thus assign the cause to a *preceding*, and the effect to a *succeeding* time.

I may observe further, as a corollary from what has been said, that the measure of time is different, when we consider different kinds of causation; and in each case, is *homogeneous* with the changes which causation effects. In the consideration of mechanical causes, we measure time by mechanical changes;—by uniform motion, or uniform succession of cycles of motion; by the rotation of a wheel, or the oscillation of a pendulum. But if we have to consider physiological changes, the progress of time is physiologically measured;—by the normal progress of vital operations; by the circulation, digestion or developement of the organized body; by the pulse, or by the growth. These different measures of time give to time, so far as it is exhibited by facts and events, a different character in the different cases. Phenomenal time has a different nature and essence according to the kind of the changes which we consider, and which gives us our sole phenomenal indication of cause.

I fear that I am traveling into matters too abstruse and metaphysical for the occasion: but before I conclude, I will present one other aspect of the subject.

In stating the difficulty, I referred to cases of moral as well as physical causation; as when prudence produces prosperity, or when folly produces ruin. It may be asked, whether we are here to apply the same explanation;—whether we are to assume a normal condition of human existence, in which neither prudence nor folly are displayed, neither prosperity nor adversity produced;—whether we are to conceive the progress of such a state to measure the progress of time, and deviations from it to denote causes of the kind mentioned. It may be asked further, whether, if we do make this supposition, we can resolve the influence of such causes as prudence or imprudence into instantaneous acts, which produce their effects immediately: and which occupy time only by being separated by intervals of the inactive normal moral condition. To this I must here reply, that the discussion of such questions would carry me too far, and would involve speculations not included within the acknowledged domain of this Society, from which I therefore abstain. But I may say, before quitting the subject, that I do not think the suppositions above suggested are untenable; and that in order to include moral causation under the maxims of causation in general, we must necessarily make some such hypothesis. The peculiarity of that kind of causation which the will and the character exert, and which is exerted upon the will and the character, would make this case far more complex and difficult than those already considered; but, at the same time, would offer us the means of explaining what may seem harsh, in the above analogy. For instance, we should have to assume such a maxim as this: that in moral causation, time is not to be measured by the flow of mechanical or physiological events;—not by the clock, or by the pulse. Moral causation has its own clock, its own pulse, in the progress of man's moral being; and by this measure of time is the relation of moral cause and effect to be defined.

That in estimating moral causation, the progress of time is necessarily estimated by moral changes, and not by machinery,—by the progress of events, and not by the going of the clock,—is a truth familiar as a practical maxim to all who give their thoughts to dramatic or narrative fictions. Who feels any thing incongruous or extravagantly hurried in the progress of events in that great exhibition of moral causation, the tragedy of Othello? If we were asked what time those vast and terrible and complex changes of the being and feelings of the characters occupy, we should say, that, measured on its own scale, the event is of great extent;—that the transaction is of considerable magnitude in all ways. But if, with previous critics, we look into the progress of time by the day and the hour—what is the measure of this history? Forty-eight hours.

But I am going beyond the boundaries of the speculations which we usually follow in this room, and will conclude.

W. WHEWELL.

XIX. *On the Motion of a small Sphere acted upon by the Vibrations of an Elastic Medium.* *By the* Rev. JAMES CHALLIS, M.A., *Plumian Professor of Astronomy in the University of Cambridge.*

[Read *April* 26, 1841.]

IT is proposed in this Essay to give a mathematical investigation respecting the motion of a small solid sphere submitted to the dynamical action of the vibrations of a medium so constituted that the pressure (p) and density (ρ) are related to each other by the equation $p = a^2\rho$, a^2 being a certain constant.

1. For this purpose it will be convenient to obtain, first, the equations which apply to the motion of such a medium when directed to or from a centre, whether the centre be moving or stationary.

Conceive P to be a fixed point in space at which the motion of the fluid is directed to or from a moving centre C. Describe about C as a centre a spherical surface always passing through the point P, and concentric with this another passing through P', a point in CP produced. Let, at a given time t, $CP = r$, and $CP' = r'$, or $r + \delta r$, δr being supposed very small. Conceive now a conical surface, with an indefinitely small vertical angle, to have its vertex at C, and its axis coinciding with CPP', and let it *always* include the same portion (m^2) of the interior spherical surface. Then if α = the velocity of the centre C resolved in the direction of r, the radius CP at the time $t + \tau$, (τ being very small) will become $r \pm \alpha\tau$, and CP' will become $r + \delta r \pm \alpha\tau$, the interval δr being supposed not to vary with the time. Hence the portion of the exterior surface included by the conical surface at the time $t+\tau$ is $m^2 . \left(\frac{r+\delta r \pm a\tau}{r \pm a\tau}\right)^2$, or $m^2 . \left(1+\frac{\delta r}{r \pm a\tau}\right)^2$; and this, neglecting terms of the order $\delta r \times a\tau$, is equal to $\frac{m^2 r'^2}{r^2}$.

Again, let v and ρ be the velocity and density of the fluid which passed the area m^2 at the time t, and $v_,$, $\rho_,$ the values of the same quantities at any time $t+\tau$. Now the quantity of fluid which in the small time δt passes m^2 is equal to $\int m^2 \rho_, v_, d\tau$, the integral being taken from $\tau = 0$ to $\tau = \delta t$. And because

$$\rho_, v_, = \rho v + \frac{d . \rho v}{dt} \tau \text{ very nearly,}$$

this integral is equal to

$$m^2 \rho v \delta t + m^2 . \frac{d . \rho v}{dt} . \frac{(\delta t)^2}{2} + \&c.$$

Also if $v_,'$, $\rho_,'$ be the velocity and density of the fluid which is passing the area $\frac{m^2 r'^2}{r^2}$ of the exterior surface at the time $t+\tau$, the quantity of fluid which passes in the interval δt is $\int \frac{m^2 r'^2}{r^2} v_,' \rho_,' d\tau$, taken from $\tau = 0$ to $\tau = \delta t$. And, because $v_,'$ and $\rho_,'$ are what v and ρ become by very small changes of time and place,

$$v_,' \rho_,' = v\rho + \frac{d . v\rho}{dt} \tau + \frac{d . v\rho}{dr} \delta r + \&c.$$

Hence,

$$\int \frac{m^2 r'^2}{r^2} v_,' \rho_,' d\tau = \frac{m^2 r'^2}{r^2} \int \left\{ v\rho + \frac{d . v\rho}{dt} \tau + \frac{d . v\rho}{dr} \delta r + \&c. \right\} d\tau$$

$$= \frac{m^2 r'^2}{r^2} \left\{ v\rho \delta t + \frac{d . v\rho}{dt} . \frac{(\delta t)^2}{2} + \frac{d . v\rho}{dr} . \delta r \, \delta t + \&c. \right\}$$

Consequently, supposing the velocity *positive* when directed *from* the centre, the increment of matter in the space between the two areas in the time δt, is ultimately,

$$- \left\{ \frac{m^2 r'^2}{r^2} \left(v\rho + \frac{d . v\rho}{dr} \delta r \right) - m^2 v\rho \right\} \delta t;$$

$$\text{or, putting } v'\rho' \text{ for } v\rho + \frac{d \, . \, v\rho}{dr} \delta r,$$

$$- m^2 \delta t \, . \, \frac{r'^2 v' \rho' - r^2 v\rho}{r^2}.$$

Now if any point be selected between P and P', the radius to which at the time t is $r_{,}$, by what has been already shewn, the transverse section of the cone through this point at the time $t + \delta t$ is with sufficient approximation $\frac{m^2 r_{,}^2}{r^2}$, and is therefore independent of δt. Hence at any instant during the interval δt the content of the conical frustum is $\int \frac{m^2 r_{,}^2}{r^2} dr_{,}$ {from $r_{,} = r$ to $r_{,} = r'$}, or $\frac{m^2}{3r^2}(r'^3 - r^3)$. The increment of density $(\delta\rho)$ in that space in the time δt is consequently,

$$- \frac{m^2 \delta t (r'^2 \rho' v' - r^2 \rho v)}{r^2} \times \frac{3r^2}{m^2 (r'^3 - r^3)}.$$

$$\text{Hence, } \frac{\delta\rho}{\delta t} + \frac{3(r'^2 \rho' v' - r^2 \rho v)}{r'^3 - r^3} = 0;$$

and passing from differences to differentials,

$$\frac{d\rho}{dt} + \frac{d \, . \, r^2 \rho v}{r^2 dr} = 0 \; \ldots\ldots\ldots\ldots (1).$$

It is plain that since P has been assumed to be a fixed point of space, the differential coefficients here are partial. The above equation, with

$$p = a^2 \rho \; \ldots\ldots\ldots\ldots (2),$$

and that derived from D'Alembert's Principle, viz.

$$\frac{dp}{\rho dr} + \left(\frac{dv}{dt}\right) = 0 \; \ldots\ldots\ldots\ldots (3),$$

are the three equations which determine the circumstances of the motion. As the velocity (a) of the centre C in no way enters into them, we may conclude that *the same equations apply to motion tending to or from a moving centre as to motion tending to or from a fixed centre.*

2. From the equations (1), (2), (3), others more immediately applicable to the question proposed to be discussed will now be deduced.

The equation (1) is equivalent to

$$\frac{d\rho}{\rho dt} + \frac{dv}{dr} + \frac{vd\rho}{\rho dr} + \frac{2v}{r} = 0 \ldots\ldots\ldots\ldots (4);$$

and by substituting for p in (3) from (2) there results,

$$\frac{a^2 d\rho}{\rho dr} + \frac{dv}{dt} + \frac{vdv}{dr} = 0 \ldots\ldots\ldots\ldots (5).$$

If now we assume ϕ' to be such a function of r and t that the partial differential coefficient $\frac{d\phi'}{dr}$ is equal to v, and substitute this expression for v in (5), the equation is integrable with respect to r. The result is,

$$a^2 \text{ Nap. log. } \rho + \frac{d\phi'}{dt} + \frac{d\phi'^2}{2dr^2} = f(t).$$

To get rid of the arbitrary function of the time, suppose

$$\phi' = \phi + \int f(t)\, dt.$$

Then $\frac{d\phi'}{dt} = \frac{d\phi}{dt} + f(t)$, and $\frac{d\phi'}{dr} = \frac{d\phi}{dr}$. Hence

$$a^2 \text{ Nap. log. } \rho + \frac{d\phi}{dt} + \frac{d\phi^2}{2dr^2} = 0 \ldots\ldots\ldots\ldots (6)$$

Obtaining from this equation $\frac{d\rho}{\rho dt}$ and $\frac{d\rho}{\rho dr}$, substituting their values in (4), and putting $\frac{d\phi}{dr}$ for v, the result will be,

$$\frac{d^2\phi}{dr^2}\left(1 - \frac{d\phi^2}{a^2 dr^2}\right) - \frac{1}{a^2}\cdot\frac{d^2\phi}{dt^2} - \frac{2}{a^2}\cdot\frac{d\phi}{dr}\cdot\frac{d^2\phi}{drdt} + \frac{2}{r}\cdot\frac{d\phi}{dr} = 0 \ldots\ldots (7).$$

3. Before making use of this equation it will be necessary to consider the comparative values of its terms under the circumstances in which we propose to apply it. The circumstances are, that v is very small compared to a, and r always exceedingly small compared to the breadths of the waves whose dynamical action is to be investigated.

First, it is plain that the terms having a^2 in their denominators will be small compared to the others. Neglecting those terms, or, which is the same thing, considering a infinite, we have the case of an incompressible fluid, and the equation applicable to it is,

$$\frac{d^2\phi}{dr^2} + 2 . \frac{d\phi}{rdr} = 0 ; \quad \text{or} \quad \frac{d^2 . r\phi}{dr^2} = 0.$$

The integral of this equation is,

$$\phi = -\frac{f(t)}{r} + F(t).$$

$$\text{Hence } \frac{d\phi}{dr} = \frac{f(t)}{r^2}, \quad \text{and } \frac{d\phi}{dt} = -\frac{f'(t)}{r} + F'(t).$$

The known equation which gives the pressure (p) of an incompressible fluid is

$$p + \frac{d\phi}{dt} + \frac{v^2}{2} = 0.$$

Hence by substitution,

$$p = \frac{f'(t)}{r} - \frac{v^2}{2} - F'(t).$$

As this equation contains two arbitrary functions, two conditions of the motion may be arbitrarily assumed. Let us assume for one condition, that the excess of the pressure (p) above the pressure Π, which would exist in the undisturbed state of the fluid, is solely owing to a velocity arbitrarily impressed in the direction of r. Then v and $f'(t)$ being supposed to vanish when $p = \Pi$, we must have,

$$p - \Pi = \frac{f'(t)}{r} - \frac{v^2}{2}.$$

As a second condition, let us suppose that the velocity is impressed at a given distance (r), and is given at any time t by the expression $m \sin bt$. Hence $f(t) = mr^2 \sin bt$, and $f'(t) = bmr^2 \cos bt$. Consequently by substituting,

$$p - \Pi = mbr \cos bt - \frac{m^2}{2} \sin^2 bt,$$

an exact equation, which gives the pressure at the distance r at any time. The two terms will be of the same order if m be not very small compared to $2br$.

Next, let us try the effect of retaining one of the omitted terms of the equation (7) and neglecting the others. Retaining, first, the term $\frac{d^2\phi}{dr^2} \times \frac{d\phi^2}{a^2 dr^2}$, and putting v for $\frac{d\phi}{dr}$, we shall have,

$$\frac{dv}{dr}\left(1 - \frac{v^2}{a^2}\right) + \frac{2v}{r} = 0,$$

$$\text{or } \frac{dv}{vdr} - \frac{vdv}{a^2 dr} + \frac{2}{r} = 0.$$

Integrating with respect to r,

$$\text{Nap. log. } vr^2 - \frac{v^2}{2a^2} = \text{Nap. log.} f(t);$$

$$\therefore \; vr^2 = f(t)\, e^{\frac{v^2}{2a^2}} = f(t)\,\left(1 + \frac{v^2}{2a^2} + \&\text{c.}\right)$$

Hence neglecting terms removed in order by two degrees from those retained, $v = \frac{f(t)}{r^2}$. This is the same result as in the case of the incompressible fluid, and by reasoning in the same manner as for that case, it will be found from equation (6) that

$$a^2 \text{ Nap. log. } \rho = \frac{f'(t)}{r} - \frac{v^2}{2} - F'(t).$$

If $\rho = 1 + \sigma$, and we assume that the value of σ depends only on a disturbance in the direction of r, it will follow that $F'(t) = 0$, v and $f'(t)$ being supposed to vanish when $\rho = 1$.

$$\text{Hence } \rho = e^{\frac{f'(t)}{a^2 r} - \frac{v^2}{2a^2}} = 1 + \frac{f'(t)}{a^2 r} - \frac{v^2}{2a^2}, \text{ nearly,}$$

$$\text{and } a^2\sigma = \frac{f'(t)}{r} - \frac{v^2}{2}, \text{ nearly.}$$

It appears, therefore, by the foregoing reasoning, that whenever $\frac{f'(t)}{r}$ is of the same order as v^2, the first term introduced into the expression for the pressure by the term of equation (7) which has now been considered, is of the order of v^4.

Again, let us retain the term $\frac{2}{a^2}\cdot\frac{d\phi}{dr}\cdot\frac{d^2\phi}{drdt}$, rejecting the others. Then

$$\frac{d^2\phi}{dr^2}-\frac{2}{a^2}\cdot\frac{d\phi}{dr}\cdot\frac{d^2\phi}{drdt}+\frac{2d\phi}{rdr}=0,$$

or $$\frac{dv}{vdr}-\frac{2}{a^2}\cdot\frac{d^2\phi}{drdt}+\frac{2}{r}=0.$$

Hence, integrating with respect to r,

$$\text{Nap. log. } vr^2-\frac{2}{a^2}\cdot\frac{d\phi}{dt}=\text{Nap. log.} f(t);$$

$$\therefore\ vr^2=f(t)\,.\,e^{\frac{2}{a^2}\cdot\frac{d\phi}{dt}}=f(t)\,.\left(1+\frac{2}{a^2}\cdot\frac{d\phi}{dt}\right),\ \text{nearly.}$$

Under the same conditions as in the last case, the first approximations to the value of v and $\frac{d\phi}{dt}$ are $\frac{f(t)}{r^2}$ and $-\frac{f'(t)}{r}$. Substituting this latter quantity in the above equation,

$$\frac{d\phi}{dr}=\frac{f(t)}{r^2}-\frac{2f(t).f'(t)}{a^2r^3},$$

and integrating with respect to r, without adding an arbitrary function of the time,

$$\phi=-\frac{f(t)}{r}+\frac{f(t)f'(t)}{a^2r^2};$$

$$\therefore\ \frac{d\phi}{dt}=-\frac{f'(t)}{r}+\frac{\{f'(t)\}^2+f(t).f''(t)}{a^2r^2}.$$

To take a particular instance, let the velocity impressed at the time t at the distance r from the centre, and in the direction of this radius, be $m\sin\frac{2\pi at}{\lambda}$, a being supposed very large compared to m, and λ very large compared to r. Then, for first approximations,

$$f(t)=mr^2\sin\frac{2\pi at}{\lambda},\ f'(t)=\frac{2\pi amr^2}{\lambda}\cos\frac{2\pi at}{\lambda},\ \text{and}\ f''(t)=-\frac{4\pi^2a^2mr^2}{\lambda^2}\sin\frac{2\pi at}{\lambda}.$$

These values will enable us to estimate the order of the second term of the above expression for $\frac{d\phi}{dt}$. By substitution they give for this term,

$\frac{4\pi^2 m^2 r^2}{\lambda^2} \cos \frac{4\pi at}{\lambda}$, which, with regard to a^2 is of the order of $\frac{m^2}{a^2} \times \frac{r^2}{\lambda^2}$, that is, of the fourth order; whilst the first term in the expression for $\frac{d\phi}{dt}$ is with regard to a of the order of $\frac{m}{a} \times \frac{r}{\lambda}$, that is, of the second order. Hence we may conclude as before that the first term introduced into the expression for the pressure by the term of equation (7) just considered, is of the *fourth* order.

4. Lastly, let us retain the term $-\frac{1}{a^2} \cdot \frac{d^2\phi}{dt^2}$ of equation (7), and reject the other small terms. We shall then have,

$$\frac{d^2\phi}{dr^2} - \frac{1}{a^2} \cdot \frac{d^2\phi}{dt^2} + \frac{2}{r} \cdot \frac{d\phi}{dr} = 0; \quad \text{or} \quad \frac{d^2 . r\phi}{dt^2} = a^2 . \frac{d^2 . r\phi}{dr^2}.$$

The known integral of this equation is

$$r\phi = f(r - at) + F(r + at).$$

The second arbitrary function applies to a disturbance which causes propagation *towards* the centre; and as such a motion is excluded by the nature of the question to the solution of which the present reasoning is directed, I shall suppose this function to vanish. Then,

$$\phi = \frac{f(r - at)}{r}.$$

$$\frac{d\phi}{dt} = -a . \frac{f'(r - at)}{r}.$$

$$\frac{d\phi}{dr} = \frac{f'(r - at)}{r} - \frac{f(r - at)}{r^2}.$$

As an application of this solution, let us suppose the velocity impressed at any time t in the direction of r, and at the distance r from the centre, to be $m\phi(t)$. Then, putting for shortness' sake u for $f(r - at)$, we shall have

$$m\phi(t) = -\frac{1}{ar} \cdot \frac{du}{dt} - \frac{u}{r^2},$$

$$\text{or} \quad \frac{du}{dt} + \frac{a}{r} . u + mar\phi(t) = 0,$$

an equation in which u and t are the only variables, and which serves to determine the value of the function $f(r - at)$ from the given value of $\phi(t)$. This equation gives by integration,

$$u = Ce^{-\frac{at}{r}} - mare^{-\frac{at}{r}} \int e^{\frac{at}{r}} \phi(t)\,dt,$$

$$\frac{du}{dt} = -\frac{Ca}{r} e^{-\frac{at}{r}} - mare^{-\frac{at}{r}} \int e^{\frac{at}{r}} \phi'(t)\,dt.$$

Now $\frac{du}{dt} = -af'(r - at) = r\frac{d\phi}{dt}$. Therefore

$$\frac{d\phi}{dt} = -\frac{Ca}{r^2} e^{-\frac{at}{r}} - mae^{-\frac{at}{r}} \int e^{\frac{at}{r}} \phi'(t)\,dt \ldots\ldots\ldots\ldots (8),$$

As an example of the application of this formula, let us suppose as before that $\phi(t) = \sin\frac{2\pi at}{\lambda}$. Then $\phi'(t) = \frac{2\pi a}{\lambda}\cos\frac{2\pi at}{\lambda}$. Also

$$\int e^{\frac{at}{r}} \cos\frac{2\pi at}{\lambda}\,dt = \frac{e^{\frac{at}{r}}\left(\frac{a}{r}.\cos\frac{2\pi at}{\lambda} + \frac{2\pi a}{\lambda}\sin\frac{2\pi at}{\lambda}\right)}{\frac{a^2}{r^2} + \frac{4\pi^2 a^2}{\lambda^2}}$$

$$= \frac{\lambda e^{\frac{at}{r}}}{2\pi a}.\sin\alpha\cos\left(\frac{2\pi at}{\lambda} - \alpha\right),$$

by substituting $\tan\alpha$ for $\frac{2\pi r}{\lambda}$. Hence

$$\frac{d\phi}{dt} = -\frac{Ca}{r^2} e^{-\frac{at}{r}} - ma\sin\alpha\cos\left(\frac{2\pi at}{\lambda} - \alpha\right).$$

and consequently by equation (6),

$$a^2 \text{ Nap. log. } \rho = \frac{Ca}{r^2} e^{-\frac{at}{r}} + ma\sin\alpha\cos\left(\frac{2\pi at}{\lambda} - \alpha\right) - \frac{m^2}{2}\sin^2\frac{2\pi at}{\lambda} \ldots\ldots (9).$$

It will be seen by the above result that the term of equation (7), retained in this instance, introduces into the expression for ρ a quantity of the order of $\frac{m}{a} \times \sin^2\alpha$, or of $\frac{m}{a} \times \frac{r^2}{\lambda^2}$, and therefore of the *third* order. Hence that term is more considerable than the other small terms of equation (7), and we may be confident that by retaining it and rejecting the others, our approximation will be exact to at least terms of the *second* order.

To determine the arbitrary constant in the equation above, let us suppose that when $t = 0$, $\rho = 1$. Hence

$$\frac{Ca}{r^2} + ma \sin \alpha \cos \alpha = 0;$$

and $$a^2 \text{ Nap. log. } \rho = ma \sin \alpha \left\{ -\cos \alpha \, e^{-\frac{at}{r}} + \cos \left(\frac{2\pi at}{\lambda} - \alpha \right) \right\} - \frac{m^2}{2} \sin^2 \frac{2\pi at}{\lambda}.$$

It is worthy of remark, that if we confine ourselves to quantities of the second order, the above result coincides with that obtained in Art. 3, for an incompressible fluid. For to that degree of approximation $\sin \alpha = \frac{2\pi r}{\lambda}$; and if $\rho = 1 + \sigma$, $a^2 \text{ Nap. log. } \rho = a^2 \sigma$. Also the term involving $e^{-\frac{at}{r}}$ will disappear after a very short time on account of the great magnitude of $\frac{a}{r}$. Hence

$$a^2 \sigma = m \cdot \frac{2\pi a}{\lambda} \cdot r \cos \frac{2\pi at}{\lambda} - \frac{m^2}{2} \sin^2 \frac{2\pi at}{\lambda},$$

which, by putting b for $\frac{2\pi a}{\lambda}$, evidently coincides with the result obtained in Art. 3.

5. Prior to the consideration of the dynamical action of the fluid in vibration on a small sphere, it will be convenient to determine the pressure on the surface of a small sphere performing small rectilinear vibrations in the fluid at rest.

The sphere is supposed to be perfectly smooth, and therefore incapable of impressing motion on the fluid in directions perpendicular to the radii. Hence the motion given to the fluid by the motion of the sphere is directed to or from a moving centre. If V be the velocity of the sphere at the time t, and θ the angle which a radius to any point of the surface makes with the straight line in which the centre is moving, $V \cos \theta$ is the velocity impressed on the fluid at that point at the same time. Now as this normal velocity varies at a given instant from one point to another of the surface, it follows that there will also be variation of density. The effect of this variation of density will be to cause the motion of each particle in contact with the spherical surface to be *curvilinear*, and to be continually directed to or from the varying positions of the centre*. Hence the equation (1) obtained in Article (1) will be applicable to the case before us by merely substituting $V\cos\theta$ for v. The same substitution being made in equation (3), the three equations (1), (2), (3) may be immediately made use of for our present purpose.

Let $V = m\phi(t)$. Then (8) becomes for this case,

$$\frac{d\phi}{dt} = -\frac{Ca}{r^2}e^{-\frac{at}{r}} - am\cos\theta e^{-\frac{at}{r}}\int e^{\frac{at}{r}}\phi'(t)\,dt,$$

and from equation (6),

$$a^2 \text{ Nap. log. } \rho = \frac{Ca}{r^2}e^{-\frac{at}{r}} + am\cos\theta e^{-\frac{at}{r}}\int e^{\frac{at}{r}}\phi'(t)\,dt - m^2\cos^2\theta\,\overline{\phi(t)}|^2.$$

Suppose that $\int e^{\frac{at}{r}}\phi'(t)\,dt = e^{\frac{at}{r}}\psi(t)$, and that $\rho = 1$, $\phi(t) = 0$, and $\psi(t) = k$, when $t = 0$. Hence,

$$0 = \frac{Ca}{r^2} + kam\cos\theta,$$

and $$a^2 \text{ Nap. log. } \rho = am\cos\theta\left\{\psi(t) - ke^{-\frac{at}{r}}\right\} - \frac{m^2}{2}\cos^2\theta\,\overline{\phi(t)}|^2. \quad (10.)$$

If we put $1 + \sigma$ for ρ and neglect terms of the order of σ^2, we shall have a^2 Nap. log. $\rho = a^2\sigma =$ the effective pressure on a unit of surface of the sphere. The effective pressure on the whole sphere estimated in the *positive* direction of the sphere's motion is $-2\pi r^2\int a^2\sigma\sin\theta\cos\theta\,d\theta$, taken from $\theta = 0$ to $\theta = \pi$. The negative sign is prefixed because it has been already assumed that the velocity of the fluid is positive when it tends *from* a centre, and as the central velocity in this instance is $m\cos\theta\phi(t)$, θ must be measured from the point of the sphere which is foremost when the motion is in the *positive* direction, so that the resultant of the pressure on an annulus of breadth $rd\theta$ and radius $r\sin\theta$ is in the negative direction when $\cos\theta$ is positive.

* See the proof of this assertion in the 'Note' added to this paper.

Now since $\int_\pi^0 \sin\theta \cos^2\theta\, d\theta = \frac{2}{3}$, and $\int_\pi^0 \sin\theta \cos^3\theta\, d\theta = 0$, the whole resulting pressure on the sphere is

$$-\frac{4\pi a m r^2}{3}\left\{\psi(t) - k e^{-\frac{at}{r}}\right\}.$$

And if δ be the ratio of the density of the fluid to that of the sphere, the accelerative force of the resistance of the fluid is

$$-\frac{am\delta}{r}\left\{\psi(t) - k e^{-\frac{at}{r}}\right\},$$

which on account of the very small factor $e^{-\frac{at}{r}}$ will after a short interval become,

$$-\frac{am\delta}{r}\cdot\psi(t).$$

6. Suppose, for example, the sphere to vibrate as a pendulum, and the extent of the vibrations to be so small that the motion of the centre may be considered rectilinear. Let l be the length of the pendulum, and x the distance of the centre of the sphere at the time t from the position it would have at rest. Then, taking the buoyancy of the fluid into account, we have for the accelerative force of gravity $-\frac{gx}{l}(1-\delta)$; and consequently, by the foregoing reasoning,

$$\frac{d^2x}{dt^2} = -\frac{gx}{l}(1-\delta) - \frac{am\delta}{r}e^{-\frac{at}{r}}\left\{\int e^{\frac{at}{r}}\cdot\frac{d^2x}{m\,dt^2}dt - k\right\}.$$

Now $\int e^{\frac{at}{r}}\frac{d^2x}{dt^2}dt = e^{\frac{at}{r}}\left(\frac{r}{a}\cdot\frac{d^2x}{dt^2} - \frac{r^2}{a^2}\cdot\frac{d^3x}{dt^3}\right)$ very nearly. Hence, by substituting,

$$\frac{d^2x}{dt^2} = -\frac{gx}{l}\cdot\frac{1-\delta}{1+\delta} + \frac{r\delta}{a(1+\delta)}\cdot\frac{d^3x}{dt^3} + \frac{kam\delta}{r(1+\delta)}e^{-\frac{at}{r}} \;\ldots\ldots (11).$$

Hence, for a first approximation, after a very small time,

$$\frac{d^2x}{dt^2} = -\frac{gx}{l}\cdot\left(\frac{1-\delta}{1+\delta}\right).$$

This equation not containing a is true of an incompressible fluid. It is, in fact, when applied to this case, an exact equation, as appears from the reasoning in the Cambridge Philosophical Transactions, Vol. V. Part II. p. 200. By equating the factor in brackets to $1-n\delta$, it will be seen that $n=\frac{2}{1+\delta}$, which when δ is very small is nearly equal to 2*.

Before proceeding to a second approximation let us determine the value of the constant k. This will be obtained by finding the value of $\frac{r}{a}\cdot\frac{d^2x}{mdt^2}-\frac{r^2}{a^2}\cdot\frac{d^3x}{mdt^3}$ when $t=0$. Now from (11), neglecting the term involving $\frac{r}{a}$ as a factor, and differentiating,

$$\frac{d^3x}{dt^3}=-\frac{g}{l}\cdot\frac{dx}{dt}\cdot\frac{1-\delta}{1+\delta}-\frac{ka^2m\delta}{r^2(1+\delta)}e^{-\frac{at}{r}};$$

and from the same equation,

$$\frac{d^2x}{dt^2}-\frac{r}{a}\cdot\frac{d^3x}{dt^3}=-\frac{gx}{l}\cdot\frac{1-\delta}{1+\delta}-\frac{r}{a(1+\delta)}\cdot\frac{d^3x}{dt^3}+\frac{kam\delta}{r(1+\delta)}e^{-\frac{at}{r}}.$$

Hence, supposing $x=h$ and $\frac{dx}{dt}=0$ when $t=0$, we readily obtain,

$$\frac{kam}{r}=-\frac{gh}{l}\cdot\frac{1-\delta}{1+\delta}+\frac{kam\delta}{r(1+\delta)^2}+\frac{kam\delta}{r(1+\delta)}.$$

Whence,

$$\frac{kam}{r(1+\delta)}=-\frac{gh}{l}\cdot(1-\delta).$$

Substituting now this value in the approximate expression for $\frac{d^3x}{dt^3}$, we get

$$\frac{d^3x}{dt^3}=-\frac{g}{l}\cdot\frac{dx}{dt}\cdot\frac{1-\delta}{1+\delta}+\frac{gha\delta}{rl}\cdot(1-\delta)e^{-\frac{at}{r}},$$

* I have already obtained this result in the London and Edinburgh Philosophical Magazine for September 1833 (p. 186), in the Cambridge Philosophical Transactions as above cited, and more recently in the Philosophical Magazine for December 1840 (p. 461). The reasoning in the last of these solutions, not embracing those terms involving the square of the velocity which may be of equal magnitude with terms retained, cannot be considered so complete as that I have now given.

and consequently by (11),

$$\frac{d^2x}{dt^2} = -\frac{gx}{l}\cdot\frac{1-\delta}{1+\delta} - \frac{gr}{la}\cdot\frac{(1-\delta)\delta}{(1+\delta)^2}\cdot\frac{dx}{dt} - \frac{gh\delta}{l}\cdot\frac{1-\delta}{1+\delta}\cdot e^{-\frac{at}{r}}.$$

Putting for shortness' sake n^2 for $\frac{g}{l}\cdot\frac{1-\delta}{1+\delta}$, we shall have

$$\frac{d^2x}{dt^2} + \frac{n^2r\delta}{a(1+\delta)}\cdot\frac{dx}{dt} + n^2x = -n^2h\delta e^{-\frac{at}{r}}.$$

By integrating this equation and neglecting terms involving $\frac{r^2}{a^2}$, which will be wholly insignificant, it will be found that

$$\frac{dx}{dt} = -hne^{-\frac{n^2r\delta t}{2a(1+\delta)}}.\sin nt,$$

by means of which equation the decrements of the successive arcs of vibration may be calculated. It is remarkable that for an incompressible fluid, for which a is infinitely great, there is no decrement of the arcs excepting so far as it arises from friction and capillary attraction. The index of e in the equation above is too small to account for the observed decrements in air, which must be mainly owing to friction.

7. As another example, let the velocity $= m\beta(1-e^{-\gamma t}) + m\phi(t)$ Then

$$e^{\frac{at}{r}}\psi(t) = \int e^{\frac{at}{r}}\{\beta\gamma e^{-\gamma t} + \phi'(t)\}\,dt = \frac{\beta\gamma}{\frac{a}{r}-\gamma}\,e^{\left(\frac{a}{r}-\gamma\right)t} + \int e^{\frac{at}{r}}\phi'(t)\,dt.$$

Hence $\psi(t) = \dfrac{\beta\gamma e^{-\gamma t}}{\frac{a}{r}-\gamma} + e^{-\frac{at}{r}}\displaystyle\int e^{\frac{at}{r}}\phi'(t)\,dt,$

and the accelerative force of the resistance is

$$-\frac{am\delta}{r}\cdot\left\{\frac{\beta\gamma}{\frac{a}{r}-\gamma}e^{-\gamma t} + e^{-\frac{at}{r}}\int e^{\frac{at}{r}}\phi'(t)\,dt - ke^{-\frac{at}{r}}\right\}.$$

If γ be an exceedingly large quantity, in which case the sphere's velocity after a very short interval is $m\{\beta+\phi(t)\}$, the above result becomes for all values of t which are not exceedingly small,

$$-\frac{am\delta}{r} \cdot e^{-\frac{at}{r}} \int e^{\frac{at}{r}} \phi'(t)\, dt,$$

and therefore the same as if the velocity were simply $m\phi(t)$. Hence a small sphere moving with a uniform velocity suffers *no resistance;* and if its velocity be partly uniform and partly variable, the resistance depends only on the variable part.

9. I come now to the consideration of the motion of a small sphere supposing it acted upon by the pressure resulting from a series of vibrations of the fluid, no other force acting. I suppose the vibrations to be propagated with the uniform velocity a in the *positive* direction, and the velocity of the vibrating fluid to be $m\phi(t)$ at the origin of x at any time t. At the same time t at any distance x from the origin the velocity is $\phi\left(t - \frac{x}{a}\right)$, being that which was at the origin at the time $\left(t - \frac{x}{a}\right)$. Suppose the centre of the sphere to be at the origin of x when $t = 0$, and to be at the distance x at the time t. For the sake of simplicity I shall first assume the vibrations of the fluid to be unaccompanied by change of density, which is a supposable case if we conceive all the parts of the fluid to move in the direction of x at the same time with the same velocity. Now it is clear that the action of the fluid on the sphere depends only on the *difference* of their velocities. And the mathematical conditions of the question will remain the same if we suppose the fluid to be at rest and the sphere to have the velocity $-\left\{m\phi\left(t - \frac{x}{a}\right) - \frac{dx}{dt}\right\}$. Calling this velocity $-m\chi(t)$, and f the resulting accelerative force of the sphere, we shall have by what was proved in Art. 5,

$$f = \frac{am\delta}{r} \cdot \left\{e^{-\frac{at}{r}} \int e^{\frac{at}{r}} \chi'(t)\, dt - ke^{-\frac{at}{r}}\right\}.$$

Let us now suppose the fluid vibrations to be accompanied by change of density. If ρ_1 be the density where the velocity is $m\phi\left(t - \frac{x}{a}\right)$, it is well known that we have the exact relation $\rho_1 = e^{\frac{m}{a}\phi\left(t - \frac{x}{a}\right)}$. The velocity of the fluid being supposed to remain the same as before, the effect of change of density will be taken into account by merely substituting $\rho_1\delta$ for δ in the equation above. By this substitution let f become f'. Then

$$f' = \frac{am\delta}{r} e^{\frac{m}{a}\chi(t) + \frac{dx}{adt}} \left\{ e^{-\frac{at}{r}} \int e^{\frac{at}{r}} \chi'(t)\, dt - ke^{-\frac{at}{r}} \right\}.$$

Again, the sphere will be acted upon by an additional accelerative force arising from the circumstance that the density, and consequently the pressure, varies from one point to another of its surface at a given time, on account of the variation of density of the fluid in vibration with the distance x from the origin at a given time. The pressure at all points of a plane perpendicular to the direction of x will evidently be the same. Hence, if $a^2 f(x)$ represent the pressure at any distance x, corresponding to the position of the centre of the sphere, it may readily be shewn that the accelerative force in question is $-a^2\delta \frac{d.f(x)}{dx}$, terms involving r^2 being omitted. Now

$$f(x) = e^{\frac{m}{a}\phi\left(t - \frac{x}{a}\right)}, \text{ and } \frac{d.f(x)}{dx} = -\frac{m}{a^2} . e^{\frac{m}{a}\phi\left(t - \frac{x}{a}\right)} \times \phi'\left(t - \frac{x}{a}\right).$$

Hence, calling this force f'', we have,

$$f'' = m\delta e^{\frac{m}{a}\chi(t) + \frac{dx}{adt}} \left\{ \phi'\left(t - \frac{x}{a}\right) \right\}.$$

As the sphere is solicited by no other forces than those just considered, $f' + f'' = \frac{d^2x}{dt^2}$; and consequently,

$$\frac{d^2x}{dt^2} = m\delta e^{\frac{m}{a}\chi(t) + \frac{dx}{adt}} \left\{ \phi'\left(t - \frac{x}{x}\right) + \frac{a}{r} e^{-\frac{at}{r}} \int e^{\frac{at}{r}} \chi'(t)\, dt - \frac{ka}{r} e^{-\frac{at}{r}} \right\} \ldots\ldots (12).$$

$$\text{Now } \int e^{\frac{at}{r}} \chi'(t)\, dt = e^{\frac{at}{r}} . \frac{r}{a} . \left\{ \chi'(t) - \frac{r}{a} \chi''(t) \right\}, \text{ nearly:}$$

$$\text{and } e^{\frac{m}{a}\chi(t)} = e^{\frac{m}{a}\phi\left(t - \frac{x}{a}\right) - \frac{dx}{adt}} = e^{\frac{m}{a}\phi\left(t - \frac{x}{a}\right)} \times \left(1 - \frac{dx}{adt}\right), \text{ nearly. Hence}$$

$$e^{-\frac{dx}{adt}} . \frac{d^2x}{dt^2} = m\delta . e^{\frac{m}{a}\phi\left(t - \frac{x}{a}\right)} \phi'\left(t - \frac{x}{a}\right)\left(1 - \frac{dx}{adt}\right) + m\delta e^{\frac{m}{a}\chi(t)} \left\{ \chi'(t) - \frac{r}{a} \chi''(t) \right\}$$
$$- \frac{kam\delta}{r} e^{\frac{m}{a}\chi(t)} e^{-\frac{at}{r}}.$$

Consequently, by integrating to the second approximation,

$$c - a \, . \, e^{-\frac{dx}{adt}} = a\delta e^{\frac{m}{a}\phi\left(t-\frac{x}{a}\right)} + a\delta e^{\frac{m}{a}\chi(t)} - \frac{mr\delta}{a}\chi'(t) + km\delta e^{-\frac{at}{r}} + \frac{km^2\delta}{a}e^{-\frac{at}{r}}\chi(t).$$

When x, t, and $\dfrac{dx}{dt}$ each $= 0$,

$$c - a = 2a\delta - \frac{mr\delta}{a}\chi'(0) + km\delta,$$

and to the same degree of approximation $k = \dfrac{r}{a}\chi'(0)$. Hence $c = a + 2a\delta$. Therefore, when t is not exceedingly small,

$$a + 2a\delta - ae^{-\frac{dx}{adt}} = a\delta\left(e^{\frac{m}{a}\phi\left(t-\frac{x}{a}\right)} + e^{\frac{m}{a}\chi(t)}\right) - \frac{mr\delta}{a}\chi'(t),$$

and expanding the exponentials to terms containing m^2,

$$\frac{dx}{dt} - \frac{dx^2}{2adt^2} = m\delta\left\{\phi\left(t - \frac{x}{a}\right) + \chi(t)\right\} + \frac{m^2\delta}{2a}\left\{\overline{\phi\left(t-\frac{x}{a}\right)}\Big|^2 + \overline{\chi(t)}\Big|^2\right\} - \frac{mr\delta}{a}\chi'(t).$$

Hence $\dfrac{dx}{dt} = \dfrac{2m\delta}{1+\delta}\phi\left(t - \dfrac{x}{a}\right)$, for a first approximation.

Therefore $\chi(t) = \phi\left(t - \dfrac{x}{a}\right)\left(1 - \dfrac{2\delta}{1+\delta}\right) = \phi\left(t - \dfrac{x}{a}\right) . \dfrac{1-\delta}{1+\delta}$,
to the same approximation.

And $\chi'(t) = \phi'\left(t - \dfrac{x}{a}\right) . \dfrac{1-\delta}{1+\delta} . \left(1 - \dfrac{dx}{adt}\right) = \phi'\left(t - \dfrac{x}{a}\right) . \dfrac{1-\delta}{1+\delta}$, nearly.

$$\text{Hence } \frac{dx}{dt}(1+\delta) = 2m\delta\phi\left(t - \frac{x}{a}\right) + \frac{m^2\delta}{2a}\left[\overline{\phi\left(t-\frac{x}{a}\right)}\Big|^2 . \left\{1 + \left(\frac{1-\delta}{1+\delta}\right)^2\right\}\right]$$
$$+ \frac{2m^2\delta^2}{a(1+\delta)^2} . \overline{\phi\left(t-\frac{x}{a}\right)}\Big|^2$$
$$- \frac{mr\delta}{a} . \frac{1-\delta}{1+\delta} . \phi'\left(t - \frac{x}{a}\right).$$

And finally,

$$\frac{dx}{dt} = \frac{2m\delta}{1+\delta} . \phi\left(t - \frac{x}{a}\right) - \frac{mr\delta(1-\delta)}{a(1+\delta)^2}\phi'\left(t - \frac{x}{a}\right) + \frac{m^2\delta}{a(1+\delta)} . \overline{\phi\left(t-\frac{x}{a}\right)}\Big|^2.$$

I shall content myself in the present communication with having obtained this equation, which, as far as I am aware, is the first instance of a solution of a problem of this kind. On a future occasion I propose making some applications of it. I shall here only remark, in confirmation of the result, that if $\phi\left(t-\frac{x}{a}\right)=1$, that is, if the fluid move with the uniform velocity m, $\frac{dx}{dt}=\frac{2m\delta}{1+\delta}+\frac{m^2\delta}{a(1+\delta)}$; and supposing the density of the sphere to be the same as that of the fluid in motion, and consequently $\delta=e^{-\frac{m}{a}}$, it will be found that $\frac{dx}{dt}=m$, neglecting m^3, &c. This manifestly should be the case. If

$$\phi\left(t-\frac{x}{a}\right)=\sin\frac{2\pi}{\lambda}(at-x),$$

the last term of the expression for $\frac{dx}{dt}$ will be partly constant and partly variable, and it is plain from equation (12) that the accelerative force is the same as if the constant part did not exist. If $\phi'\left(t-\frac{x}{a}\right)=0$, whenever $\phi\left(t-\frac{x}{a}\right)=0$, the sphere and the fluid will be stationary at the same instants.

J. CHALLIS.

CAMBRIDGE OBSERVATORY,
March 3, 1841.

ADDITIONAL NOTE.

THE theoretical resistance to the motion of a ball-pendulum in the air, obtained in the foregoing Essay, differs from that found by Poisson in Vol. XI. of the Memoirs of the Paris Academy of Sciences and in the *Connaissance des Tems* for 1834, and by M. Plana in a Memoir on the Motion of a Pendulum in a Resisting Medium, published at Turin in 1835. I propose therefore to add here as distinct a statement as possible of the reason of this difference.

According to the solution of these two eminent mathematicians, the motion of the fluid in contact with the oscillating sphere, is partly along its surface and partly directed to or from the centre: on the contrary, in the solution I have given, the motion at each instant is *wholly* directed to or from the centre. The following reasoning appears to prove the correctness of the latter view.

It is well known that the equation $udx + vdy + wdz = 0$, is the differential equation of a surface which cuts at right angles the directions of the motions of the particles through which it passes, if the left hand side of the equation be integrable *per se.* And if it be integrable after being multiplied by a factor N, the equation $N(udx + vdy + wdz) = 0$, is equally the differential equation of such a surface, but more general in its application. Let therefore

$$N(udx + vdy + wdz) = d.\ \phi(x, y, z, t).$$

Then integrating, and supposing the arbitrary function of the time to be included in ϕ, we shall have $\phi(x, y, z, t) = 0$. The surfaces of which this is the general equation, for the sake of shortness I shall call *surfaces of displacement.* If the time t changes to $t + dt$, the co-ordinates x, y, z, of each particle at a surface of displacement change to $x + udt$, $y + vdt$, $z + wdt$, and are ultimately the co-ordinates of the surface of displacement in a new position indefinitely near the former. Hence

$$\phi(x + udt,\ y + vdt,\ z + wdt,\ t + dt) = 0,$$

or, putting ϕ for $\phi(x, y, z, t)$,

$$\phi + \frac{d\phi}{dx} \cdot udt + \frac{d\phi}{dy} \cdot vdt + \frac{d\phi}{dz} \cdot wdt + \frac{d\phi}{dt} dt = 0.$$

Now $\phi = 0$, $\frac{d\phi}{dx} = Nu$, $\frac{d\phi}{dy} = Nv$, $\frac{d\phi}{dz} = Nw$. Hence

$$N(u^2 + v^2 + w^2) + \frac{d\phi}{dt} = 0.$$

Or, if $u^2 + v^2 + w^2 = V^2$, $N = - \frac{d\phi}{V^2 dt}$.

Let, for example, the surface of displacement be that of a sphere of given radius moving with a given velocity $V_{,}$ in the direction of the axis of z. Then if R be the radius of the sphere, and a, b, c, be the co-ordinates of its centre,

$$\phi(x, y, z, t) = (x - a)^2 + (y - b)^2 + (z - c)^2 - R^2,$$

$$V_{,} = \frac{dc}{dt}; \quad \frac{d\phi}{dt} = -2(z - c)\frac{dc}{dt} = -2V_{,}(z - c);$$

and the normal velocity V is equal to $V_{,} \cdot \frac{z - c}{R}$. Consequently $N = \frac{2R}{V_{,}(z - c)}$. It therefore appears that the surface of an oscillating sphere may be a surface of displacement, and that the factor N varies as $\frac{1}{\cos\theta}$, as I have supposed in Art. 5. It also appears that the error of Poisson's solution consists in his employing an equation depending on the supposition that $udx + vdy + wdz$ is of itself an exact differential; a supposition which, as we have seen, is not of sufficient generality. Indeed it would not be difficult to shew that this condition is fulfilled only when the surfaces of displacement coincide with *surfaces of equal pressure* during the whole of the motion, and when in consequence the motion of each particle of the fluid is *rectilinear*. The differential equations of fluid motion in their most general form have never yet been obtained.

The above considerations lead to a very simple solution of the problem of the resistance of the air to an oscillating sphere. For supposing the motion of the sphere to be impressed on the sphere and on the air in the direction contrary to that of the sphere's motion, the sphere will be reduced to rest, and the velocity of the fluid along the surface of the sphere, from what is proved above, will be $V_{,} \sin\theta$. Hence by a known Theorem of Hydro-dynamics,

$$\frac{1}{\rho} \cdot \frac{dp}{Rd\theta} + \left(\frac{d \,.\, V_{,} \sin\theta}{dt}\right) = 0;$$

and as $p = a^2\rho$, and $\frac{Rd\theta}{dt} = V_{,} \sin\theta$, if we put $m\ \phi(t)$ for $V_{,}$, it follows that,

$$\frac{a^2 d\rho}{\rho d\theta} + mR\phi'(t) \sin\theta + \{\phi(t)\}^2 \cos\theta \sin\theta = 0;$$

whence, by integration,

$$a^2 \text{ Nap. log.} \rho - R\phi'(t) \cos\theta - \frac{m^2 \{\phi(t)\}^2}{2} \cos^2\theta = \text{a function of } t.$$

This equation agrees in its ultimate application with (10) of Art. 5, and consequently leads to the same result.

CAMBRIDGE OBSERVATORY,
Dec. 13, 1841.

XX. *Description of an Extinct Lacertian Reptile,* Rhynchosaurus articeps, *Owen, of which the Bones and Foot-prints characterize the Upper New Red Sandstone at Grinsill, near Shrewsbury. By* Richard Owen, F.R.S., G.S. &c., *Hunterian Professor in the Royal College of Surgeons.*

[Read *April* 11, 1842.]

The existence of a small four-footed animal, at the period of the deposition of the New Red Sandstone near Shrewsbury, was announced by Dr. Ogier Ward of that city, at the meeting of the British Association at Birmingham; the evidence then brought forward consisting of foot-prints only. These Ichnolites most nearly resembled those figured in the Memoir on the New Red Sandstone of Warwickshire, by Messrs. Murchison and Strickland*, but differed in giving more distinct indications of the terminal claws, and less distinct impressions of the connecting web: the innermost toe is more diminutive, and there is an impression, always at a definite distance from the fore-toes, like a hind-toe pointing backwards, and which seems to have only touched the ground by its point, as in some wading birds: reminding one of the form of some of the Ichnolites discovered by Dr. Hitchcock, in the New Red Sandstone at Connecticut, which have been referred to the class of birds.

Any evidence of a warm-blooded and quick-breathing class of animals at so remote a period as the New Red Sandstone epoch requires to be very closely sifted, and the chance of obtaining any analogical facts, bearing upon the explanation of the 'Ornithicnites' of Professor Hitchcock, induced me to spare no exertions to obtain further insight into the problematical creature of the Grinsill quarries.

* Geological Transactions, Second Series, Vol. V. pl. xxviii.

Through the kind and zealous attention of Dr. Ward to the quarrying operations in his neighbourhood, various fossils were from time to time secured and transmitted to me, which at length enabled me to form a clear opinion of the nature and affinities of the animal in question. I say of the animal that impressed the sands with its feet, because, with respect to the bones, Dr. Ward observes in his letter accompanying them: "As they have always been found nearly in the same bed as that impressed by the footsteps I have described, I am induced to believe that these are the bones of the same animal:" and in this opinion, from the correspondence of size between the bones and foot-prints, and from the circumstance of the absence of other observed bones or foot-prints in the same quarry, I entirely coincide.

The vertebræ, to which my attention was first directed, proved the species to belong to the Lacertine or lower division of the great Saurian group of reptiles*. These bones will be first described.

Vertebræ.—Both surfaces of the centrum are concave and are deeper than in the biconcave vertebræ of the extinct Crocodilians; the texture of the centrum is compact throughout. In the dorsal series the two lateral surfaces join the under surface at a nearly right angle, the transverse section presenting a subquadrate form, with the angles rounded off: the under surface and sides are regularly concave longitudinally.

The neural arch is anchylosed with the centrum, without trace of suture, as in most Lizards; it immediately expands and sends outwards from

* An extended survey of the modifications of this class of Vertebrata from their first appearance on the Earth's surface to the present time, is necessarily attended with different views of their classification than can be derived from an acquaintance, however close, with existing species only. I propose to divide the Reptilia into eight orders: viz. DINOSAURIA, ENALIOSAURIA, CROCODILIA, LACERTILIA, PTEROSAURIA, CHELONIA, OPHIDIA, and BATRACHIA. They are here enumerated in the descending scale of organization. The Saurian division was represented of old by reptiles manifesting the crocodilian grade of structure, under a rich variety of modifications, constituting, besides the typical and still represented groups, two other orders, now wholly extinct; it has since subsided into a swarm of small Lacertians, headed by so few examples of the Crocodilian or Loricate species, that it is no marvel such relics of a once predominating tribe should have found a humble place in Linné's System of Nature, as co-ordinate members of the genus *Lacerta*.

each angle of its base a broad triangular process with a flat articular surface; the two anterior surfaces look directly upwards, the posterior ones downwards; the latter are continued backwards beyond the posterior extremity of the centrum; the tubercle for the simple articulation of the rib is situated immediately beneath the anterior oblique process. So far the vertebræ of the *Rhynchosaurus*, always excepting their biconcave structure, resemble the vertebræ of most recent lizards. In the modification next to be noticed, they show one of the vertebral characters of the *Dinosauria**. A broad obtuse ridge rises from the upper convex surface of the posterior articular process, and arches forwards along the neurapophysis† above the anterior articular process, and gradually subsides anterior to its base: the upper part of this arched angular ridge forms, with that of the opposite side, a platform, from the middle line of which the spinous process is developed. This structure is not present in existing lizards; the sides of the neural arch in their vertebræ immediately converge from the articular processes to the base of the spine, without the intervention of an angular ridge formed by the side of a raised platform. The base of the spinous process in the Rhynchosaur is broadest behind, and commences there by two roots or ridges, one from the upper and back part of each posterior articular process: they meet at the posterior part of the summit of the neural arch, whence the spinous process is continued upwards as a simple plate of bone, its base extending forwards along about two thirds of the length of the platform, which then again divides into two ridges which diverge from each other in slight curves to the anterior and external angles of the neurapophyses. The interspace of the diverging anterior crura of the base of the spine is occupied by a triangular fossa, not continued into the substance of the spine; this fossa is bounded below by a horizontal plate of bone extended over the anterior part of the spinal canal, and terminated by a convex outline. The anterior margin of the spinous

* The characters of this extinct Order of Reptiles are given in the Report of the British Association, 1841, p. 102.

† The vertebral nomenclature, which I have been compelled to invent for the requisite clearness and brevity of description of these most complicated and most common of Reptilian fossils, is explained in the Geological Transactions, Vol. V, pt. iii, Second Series, p. 518.

process is thin and trenchant; the height of the spine does not exceed the antero-posterior diameter of its base; it is obliquely rounded off. The spinal canal sinks into the middle part of the centrum, and rises to the base of the spine, so that its vertical diameter is twice as great at the middle as at the two extremities: this modification resembles, in a certain degree, that of the vertebræ of the *Palæosaurus* from the Bristol conglomerate*. The following are dimensions of the most perfect of the dorsal vertebræ of the *Rhynchosaurus*:—

	Lines.
The length of the centrum	$5\frac{1}{2}$
Height of the articular end	3
Breadth of the articular end	$2\frac{2}{3}$ $\frac{1}{3}$
From the lower margin of the posterior extremity of the centrum to the posterior part of the base of the spine	5
From the lower margin of the posterior extremity of the centrum to the summit of the spine	9
Antero-posterior extent of base of spine	4
Breadth of the neural arch, from the outer margin of one anterior articular process to that of the opposite side	$8\frac{1}{2}$
Breadth of the neural arch at the interspace between the anterior and posterior articular processes	4
Breadth of the neural arch across the middle of the spinous platform	2

Skull.—The most complete specimen yet obtained of this instructive part of the skeleton of the *Rhynchosaurus* is imbedded in a portion of the coarse-grained sandstone from the Grinsill quarries. The lower jaw is in its natural position, as when the mouth is shut, showing that the parts had not been dislocated when they became imbedded in the sand.

The skull presents the form of a four-sided pyramid, compressed laterally, and with the upper facet arching down in a graceful curve to the apex, which is formed by the termination of the muzzle.

The very narrow cranium, the wide temporal fossa on each side, bounded behind by the bifurcations of the parietal and the mastoid, and laterally by a strong compressed zygoma, with a long tympanic pedicle descending vertically from the point of union of the transverse and

* Geological Transactions, Second Series, Vol. V. p. 349, pl. xxix.

zygomatic arches, and terminating in a convex pulley for the articular concavity of the lower jaw,—the large and complete orbits, and the short, compressed, and bent-down maxillæ,—all combine to prove the fossil to belong to the Lacertine division of the Saurian Order.

The lateral compression and depth of the skull, the great vertical extent of the superior maxillary bone, the small relative size of the temporal spaces, the great depth of the lower jaw, prove that it does not belong to a reptile of the Batrachian Order. The shortness of the muzzle, and its compressed form, equally remove it from the Crocodilians. No Chelonian has the tympanic pedicle so long, so narrow, or so freely suspended to the posterior and lateral angles of the cranium.

The general aspect of the skull differs, indeed, from that of existing Lacertians, and singularly resembles that of the bird or turtle, and the resemblance is increased by the apparent absence of teeth. The intermaxillary bones, moreover, are double, as in the Chelonia, and also symmetrical, not united by a median ascending process; but, with this exception, all the more essential characters of the skull are those of the Lizard.

Of the proper parietes of the cerebral cavity, the portion formed by the parietal and frontal bones is exposed. The *parietal* is traversed by a thin, but high median crest longitudinally: the sides are convex, and the breadth of the bone diminishes towards the occiput: here it divides into two branches, which pass outwards, more transversely than in existing Lizards. There is no perforation either in the parietal bone, or in the coronal suture. This suture is transverse. At the anterior part of the parietal crest two lines diverge from each other at a right angle to the upper part of the orbit, and separate the median from the post-frontals; a nearly transverse suture divides the fore-part of the parietal from the post-frontals. The *median frontal* bone is single, like that of the new-world Thorictes (*Thorictes, Tejus*) and Iguanæ, not divided, as in the Varanians. It expands slightly as it advances towards the fore-part of the orbits, the oblique lines dividing the median frontal from the post-frontals, and the supra-orbitary ridges are raised, so that the interspace is slightly concave; and the surface is also broken by irregular elevations and depressions. The *post-frontal* is divided by a nearly transverse suture. This bone completes the upper and outer part of the orbit by a thin well-defined curved plate; an irregular blunt ridge descends in a nearly vertical direction behind this plate, and then the posterior frontal is continued backwards in the form of a long compressed plate gradually terminating in a point, which overlaps the zygomatic bone. This forms the medium of union between the long posterior frontal and the parietal fork. The posterior frontal is divided by a suture, as in the Iguanæ; the anterior division forms a projecting curved plate at the upper and outer part of the circumference of the orbit, and prolongs forward the rim of that cavity beyond the plane of the head.

The *tympanic* bone is bent like the Italic *ſ*, and is slightly expanded transversely at its distal extremity: its posterior surface is exposed in the present fossil, showing it to be convex and rounded, and continued externally in the form of a thin plate, which is concave posteriorly. The thick convex stem divides near the lower end into two ridges, which diverge, like the condyles of a humerus, and intercept the trochlea on which the concave articulation of the lower jaw plays. The tympanic trochlea is convex from behind forwards, concave from side to side.

The orbit is large, nearly circular in form, and its bony frame is complete; this is formed above by the median, anterior and posterior frontals, before by the anterior frontal and lachrymal, below by the malar, and behind by the malar and posterior frontal.

The *malar* bone, as in most lizards, is long, slender, and bent upon itself, but its external surface is unusually concave, the orbital plate projecting outwards, like the corresponding rim formed by the frontal bone. The anterior or horizontal branch of the malar gradually tapers to a point, which is wedged in between the lachrymal and superior maxillary; the posterior branch ascends at nearly a right angle and is applied obliquely to the posterior part of the descending process of the posterior frontal: at the angle between the two portions of the malar a process is continued backwards for about half an inch, but its extremity is broken off.

The *lachrymal* bone presents the same relative size and position as in the *Thorictes*, *Lacerta*, and most Lizards; a tubercle rises from about the middle of its external surface.

The *superior maxillary* is a broad vertical triangular plate of bone, with a smooth external surface: the alveolar border projects externally like a ridge, above which the bone is slightly concave: this ridge appears to be slightly dentated, and overlaps the corresponding alveolar border of the lower jaw. The posterior superior margin of the maxillary bone is slightly concave, and joins the malar and lachrymal bones, and a small part of the prefrontal: the anterior superior margin joins the upper half of the elongated intermaxillary bone, which divides it from the nasal bone and the external nostril: the lower side or base of the triangle, which forms the alveolar border, is convex.

The most singular character of the cranium of the present fossil Reptile is afforded by the *intermaxillary* bones. These, in their length and regular downward curvature, give to the fore-part of the skull the physiognomy of that of an accipitrine bird; but they differ essentially from both those of the bird and lizard, in being on each side distinct throughout their whole length, and in gradually diminishing to their inferior or rostral extremity, which is not expanded or continued laterally to form any part of the alveolar border of the upper jaw. Each intermaxillary bone is a slender subcompressed elongated bone, bent so as to describe a quarter of a circle; the upper half is thinner, but rather broader, than the lower one, and is wedged in between the superior maxillary, frontal, and nasal bones: the lower half, which is somewhat narrower, but thicker and subcylindrical, *projects freely downwards beyond the superior maxillary bone;* and the deep anterior extremity of the lower jaw is applied to the posterior surface of these produced extremities of the two intermaxillaries when the mouth is closed. The two intermaxillaries converge towards each other from their posterior origins, and are in close contact with each other where they form the singular curved and prominent beak.

The external nostril I presume to be situated between the upper diverging ends of the intermaxillaries, but a fracture of the fossil at this part prevents the determination of the precise form of this aperture, or the mode of termination of the nasal bones. These bones, if not actually absent in the present fossil, as in most Chelonia, must have been extremely small, as in the Chameleon.

The *lower jaw* is of considerable depth, and exceeds, as in most Saurians, the length of the cranium. The articular cavity is deep and wide: the angle of the jaw is broken off in the fossil directly behind this cavity on the left side, but is continued backwards beyond it for more than half an inch on the right side. The ramus gradually expands in the vertical direction, and becomes thinner from side to side, as it advances forwards, to about its middle part, which is just behind the orbit, where it measures 11 lines in depth: it then begins gradually to diminish vertically to the symphysis, which again slightly increases to its termination, which is obliquely truncated, much compressed laterally, and applied against the deflected extremities of the intermaxillaries. The posterior half of the maxillary ramus is slightly convex externally; the anterior narrower part slightly concave: the superior margin describes a slight but graceful sigmoid curve, convex posteriorly, and concave anteriorly, where it is adapted to the convex alveolar border of the upper maxillary bone, to the inner side of which it is closely applied. The alveolar border forms an external, convex, projecting ridge, analogous to that of the upper jaw.

The composite structure of the lower jaw is very clearly displayed in the fossil. The articular piece is short, but is continued forwards as a slender process below the angular piece *e*, as in the *Varanus*. The angular piece is relatively larger than in *Varanus*, and presents nearly the same proportions as in *Thorictes*. The supra-angular is longer, and occupies the proportion of the jaw formed by the supra-angular and coronoid elements in *Thorictes* and other Lizards. The opercular element extends farther upon the outside of the jaw from its lower margin than in the existing Lizards; *Thorictes*, again, in this respect, coming nearest to *Rhynchosaurus*. The dentary element constitutes the rest of the outer side of the ramus, but not the slightest trace of teeth is discernible. The fossil seems to have been preserved with the mouth naturally closed, and the upper and lower jaws are in close contact. In this state they must originally have been buried in the sandy matrix, which afterwards hardened around them: and since true Lizards, owing to the uninterrupted succession of their teeth, do not become edentulous by age, we must conclude that the state in which the Rhynchosaurus was buried, with its lower jaw in undisturbed articulation with the head, accorded with its natural state while living, so far as the less perishable parts of its masticatory organs were concerned. Nevertheless, since a view of the inner side of the alveolar border has not been obtained, we cannot be assured of the edentulous character of this very singular Saurian; for in the *Agamæ* and *Chameleons* the dental system, seen only from the outside of the jaws, appears to be represented by mere dentations of the alveolar border, and the anchylosed bases of the teeth, the crowns of which really form the dentations, are recognizable only by an inside view.

But the indications of the dental system are indisputably much less obvious in the *Rhynchosaurus* than in these existing Lacertians: the dentations of the upper jaw are absolutely feebler than in the Chameleon, and no trace of them can be detected in the lower jaw. The absence of the coronoid process, which is conspicuously developed in all Lizards, corresponds with the unarmed state of the jaw; and the resemblance of the *Rhynchosaurus* in this respect to the *Cheloniæ*, and to *Chelys ferox*, indicates that the correspondence actually extended to the edentulous condition of the jaws. The resemblance of the mouth to the compressed beak of certain sea-birds, the bending down of the curved and elongated intermaxillaries, so as to be opposed to the deep symphysial extremity of the lower jaw, are further indications that the ancient Rhynchosaur may have had its jaws incased by a bony sheath, as in Birds and Turtles.

I proceed now briefly to notice the other portions of the skeleton, which, from their size, texture, and community of stratum and locality, are with much probability referable to the *Rhynchosaurus*.

Considerable portions of two rami of two distinct lower jaws, in portions of sandstone from the Grinsill quarries, show the same structure as that of the jaw in the cranium above described: the thick edentulous alveolar border is bounded below on the outside by the longitudinal channel: the lower border of the ramus is thick and smoothly rounded, it is somewhat abruptly constricted immediately behind the deflected extremity or symphysis. The structure of the bone is very compact; the fractured end demonstrates the large cavity, common in Reptiles, which is included between the opercular and dentary pieces.

One piece of fine-grained sandstone contains a considerable proportion of four of the dorsal vertebræ in a connected chain, which measures 1 inch 10 lines.

Near this chain of four and a smaller part of a fifth vertebræ there are portions of four ribs. These have a simple, not a bifurcated head; they are subcompressed, pretty uniformly curved, and grooved longitudinally on both sides; the longest portion of rib measures two inches, following the curvature.

The same fragment of sandstone contains three flat bones, which offer several striking modifications, whether they be compared with the constituents of an os innominatum or of the scapular arch. The most entire of the three bones* has a thick articular end, a long, broad, and thin plate, forming the body of the bone; and a moderately long trihedral process given off from the convex margin near the articular end. In these characters the comparative anatomist conversant with the modifications of the skeleton in recent and extinct Saurians will recognise a resemblance to the scapula of the Iguanodon and Hylæosaur, in a minor degree to the ischium of the Crocodile, and somewhat more remotely to the pubis of the Tortoise. The trihedral process, in the second comparison, would match the anterior pubic process of the Crocodile's ischium, but the entire bone would differ from that of the Crocodile in the slenderness of the pubic process, in the greater breadth and less length of the body of the bone, and in its extreme thinness; it increases in thickness, however, as in the Crocodile's ischium, towards the articular end. The correspondence of the trihedral process

* Plate vi, fig. 8.

of the bone in question with the long spinous process of the Chelonian pubis, is less close than the one just discussed. If the present well-marked bone of the Rhynchosaur be regarded as a scapula, it is to that bone in the *Dinosauria* that it offers most resemblance; and the prismatic process would then correspond with the one sent off from the anterior part of the glenoid articular surface in the scapula of the Hylæosaur and Iguanodon. The concavity at the neck of the bone, at the side opposite that from which the process extends, also gives it a nearer resemblance to the Dinosaurian scapula than to the Crocodilian ischium: it differs from the scapula of the Crocodile in having the posterior margin beyond the neck straight instead of convex; the corresponding margin in the ischium being concave. The blade of the bone, considered as scapula, is broader and shorter than in either the Dinosaurs or Crocodiles: its outer surface is slightly convex. Supposing the scapula to be placed vertically upon the thicker articular end, the prismatic process is directed forwards and downwards. There are a few pits or inequalities near the neck or thick articular margin in the present fossil. The outer surface of the plate is marked with extremely fine striæ, radiating from the neck.

	In.	Lines.
Length of the bone	1	8
Breadth of the neck	0	$5\frac{1}{2}$
Breadth of the base	1	0
Length of the trihedral process	0	8

*Coracoid**.—The remains of a thin and broad plate of bone, attached by a short neck to an apparently articular thickened head or process, might be compared to a coracoid, since it resembles, so far as it is preserved, the coracoid of Lizards, more than it does any other known bone; there is not, however, the perforation near the articular surface. The breadth of the neck is 6 lines; that of the body of the bone which remains 13 lines; the length or diameter at right angles to the above is 10 lines. The bone is thinned off to an edge, which is gently convex.

Humerus†.—A third bone, imbedded in the same piece of sandstone at a little distance from the preceding, is expanded at both extremities,

* Plate vi, fig. 9, *a*. † Plate vi, fig. 9, *c*.

contracted and twisted in the middle. One of the expanded extremities, apparently the proximal end, is nearly entire: it terminates by an irregular convex border not thinned off to an edge, but adapted to the formation of a joint, and to the attachment of cartilage. The exposed surface of the expanded head is concave from side to side, somewhat resembling the expanded and bent pubic plate in Lizards. The opposite extremity is broken across: it shows the commencement of a slight longitudinal ridge near its middle part. This bone bears most resemblance to a humerus; but I am at present unable to determine it unequivocally. If compared with the left pubis of Lacertians, the entire and bent extremity corresponds with the median portion of that bone; but the middle part or stem is much longer in the fossil; and the broken end which would agree with the acetabular end of the pubis, is too thin to have entered into the formation of such a cavity in the fossil: it likewise wants the perforation which characterizes the pubis in Lizards. The same thinness and imperforate condition of the fractured end oppose the comparison of the present bone with the coracoid of the Crocodile:

	In.	Lines.
Length of this bone as far as complete	1	9
Breadth in the middle	0	3
Breadth of entire expanded extremity	0	10

In the slab containing the above-described bones there are other fragments of bone; but too small and imperfect for profitable description. Those of which I have endeavoured to make the form and analogies intelligible, though evidently peculiar, as might be expected in a Saurian with so strange a head, and perhaps with a hind toe directed backwards as in Birds, may be regarded as most probably constituents of a strong and well-developed pectoral arch, and a humerus: and they indubitably indicate a mechanism for locomotion on land, which would agree with that of the animal that has left the impressions of its footsteps upon the same sandstone.

Radius and Ulna.—Another piece of coarse-grained sandstone from the same quarry contains a series of seven or eight vertebræ, in a very fragmentary state; also two or three ribs, rather more slender and not so distinctly grooved as in the fine-grained slab, and the proximal extremities of two long bones, which most resemble a lizard's radius and ulna. The shaft of the radius is more slender than that of the ulna: one side is flat, the other convex: it expands and assumes a subtrihedral figure, by the development of a slight longitudinal ridge: its proximal end is compressed and more suddenly expanded: its breadth is $2\frac{1}{4}$ lines; that of the shaft of the bone is 1 line. The impression, partly broken away in the stone, indicates the greater expansion of the distal end of this bone, with a length of 1 inch 3 lines. The proximal end of the ulna has a distinct trihedral figure, and the expanded extremity is produced backwards, so as to indicate the olecranon: the breadth of the head is 4 lines; that of the middle of the shaft is $2\frac{1}{2}$ lines. There is a portion of a broad and flat bone in this piece which may have belonged to the scapular arch.

Ilium.—In another piece of stone, with the other portion of the same chain of five vertebræ, there is a broad flat bone, apparently terminating in a long narrow process at one end, which may be an ilium: its length is indicated to be at least 1 inch 7 lines.

Femora.—A thin piece of burr, or coarse-grained sandstone, contains the articular end of a broad and flat bone, in which the raised oblong surface of the joint is divided by a smooth channel, and may be compared with the cotyloid portion of the ilium: the same piece of stone contains the shafts of two long bones, most probably femora. The length of the most perfect of these is two inches, and this does not include the distal end: the diameter of the middle of the shaft is $2\frac{1}{2}$ lines: the surface of the preserved middle part shows the shaft to have been somewhat angular: the compact outer wall of the bone is about a quarter of a line thick: a large medullary cavity extends the whole length of the shaft, agreeing with the indications of terrestrial habits yielded by the bones before described: the extremities of the femora are spongy, but much decomposed and stained with iron-mould.

There are few genera of extinct reptiles of which it is more desirable to obtain the means of determining the precise modifications of the locomotive extremities than the *Rhynchosaurus.* The fortunate preservation of the skull has brought to light modifications of the Lacertine structure leading towards Chelonia and Birds, which were before unknown: the vertebræ, likewise, exhibit very interesting deviations from the Lacertian type. The entire reconstruction of the skeleton of the *Rhynchosaurus* may be ultimately accomplished, if the collection and preservation of the fossils of the Grinsill quarries be continued with the same activity and care, as have already produced so important an accession to Palæontology through the well-directed zeal of Dr. Ogier Ward, and other members of the Literary and Scientific Association at Shrewsbury.

DESCRIPTION OF THE PLATES.

Plate V.

Skull of the *Rhynchosaurus articeps.*

Fig. 1. Side view.
2. Three-quarters' view.
3. Upper view.
4. Front view.
5. Back view.

All the figures are of the natural size, and the same parts are marked with the same letters in each figure.

Cranium and Upper Jaw.

a Intermaxillaries.
b Nasal.
c Frontal.
d Maxillary.
e Anterior frontal.
f Lachrymal.
g Malar.
i Posterior frontal.
i' Orbital division of ditto.

Fig 1.

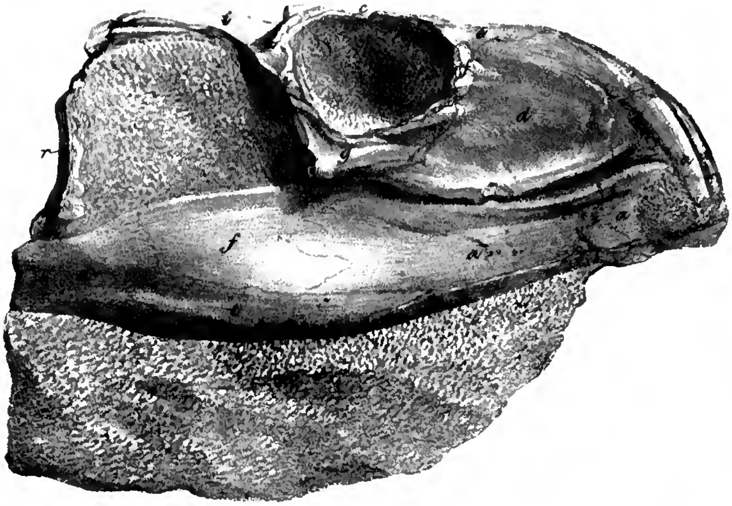


Fig 3.

Fig 4

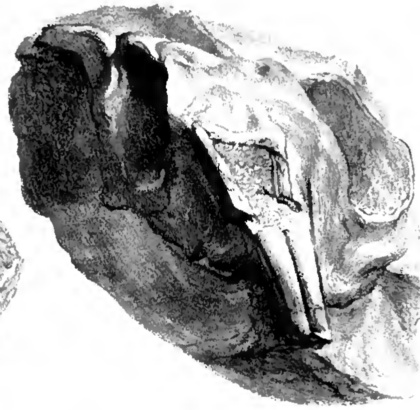


Fig 5.

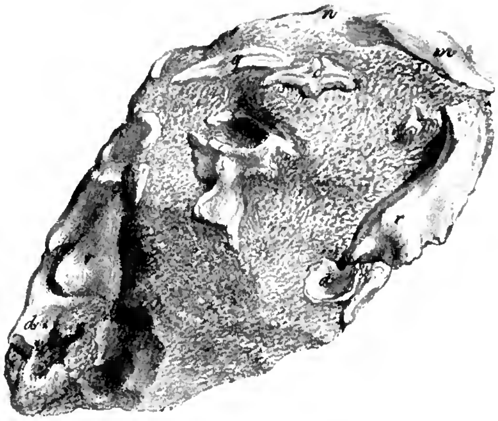
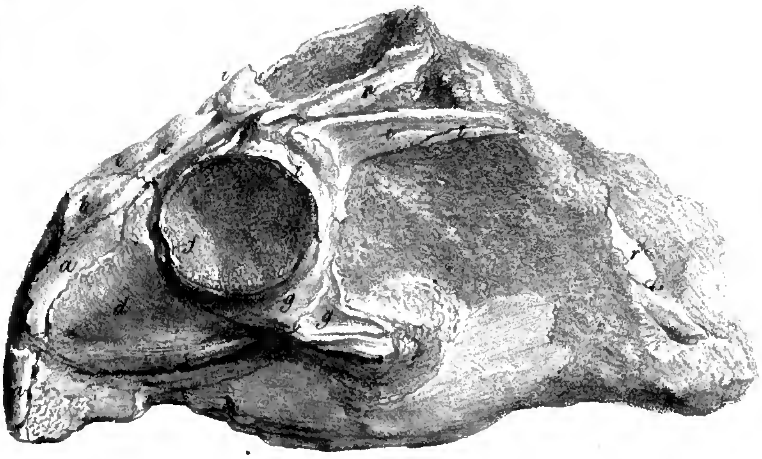
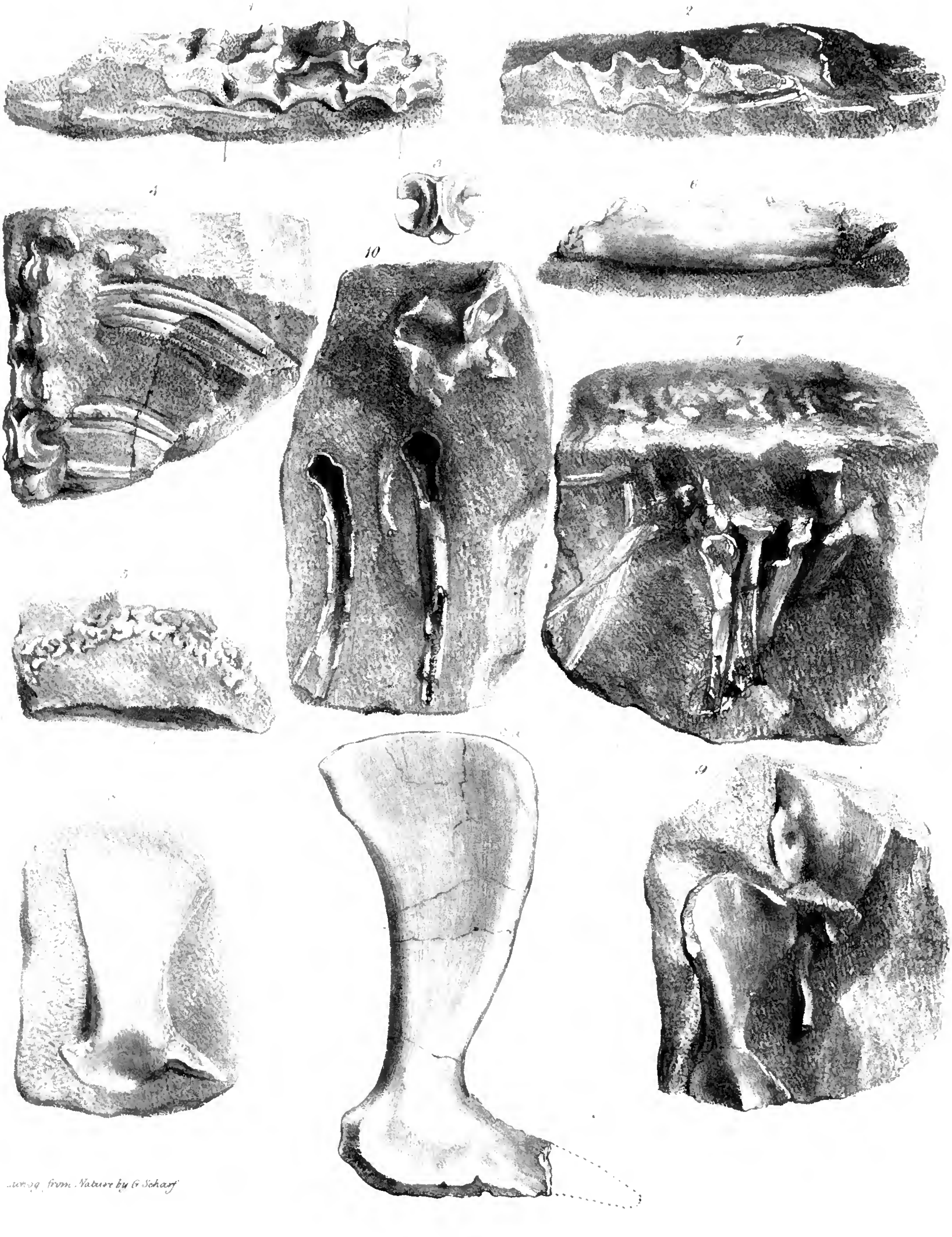


Fig 2.



Lithog. from Nature by G.Scharf

Rhynchosaurus.



Rhynchosaurus

m Mastoid.
l Temporal.
n Parietal.
o Supra-occipital.
q Bifurcated process of Parietal.
r Tympanic.

LOWER JAW.

a Dentary piece.
b Opercular ditto.
d Articular ditto.
e Angular ditto.
f Coronoid ditto.

PLATE VI.

VERTEBRÆ AND OTHER BONES OF THE *RHYNCHOSAURUS ARTICEPS*.

Fig. 1. Chain of five dorsal vertebræ, vertically bisected, shewing the concave articular surfaces, and the ventricose spinal canal.

2. Opposite and more mutilated side of the same chain, with a portion of the ilium.

3. Upper view of an entire dorsal vertebra.

4. Upper view of fig. 1, shewing the grooved ribs.

5. A chain of caudal vertebræ.

6. A portion of the left ramus of the lower jaw.

7. A chain of vertebræ, with ribs; fragment of Scapula, Radius and Ulna.

8. Right scapula.

8. *a*. Corresponding bone of the *Hylæosaurus* much reduced, shewing a similar process from the acromion.

9. Part of the coracoid *a*, the clavicle, and humerus *b*.

10. Part of the ilium, and of both femora.

All the parts of the Rhynchosaur are of the natural size.

XXI. *A general Investigation of the Differential Equations applicable to the Motion of Fluids. By the* Rev. JAMES CHALLIS, M.A., *Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge.*

[Read *April* 11, 1842.]

1. LET p be the pressure at any point of a mass of fluid in motion, the co-ordinates of which referred to three rectangular planes are x, y, z, at a time t reckoned from a given epoch; let ρ be the density at the same point and at the same time, and suppose p and ρ to be always related to each other by the equation $p = a^2\rho$. Let X, Y, Z, be the forces impressed in the directions of the three rectangular co-ordinates on the fluid particle which is at the point xyz at the time t, and let u, v, w, be the components of the velocity of the particle in the same directions. Then the two fundamental equations of Hydro-dynamics are, as is well known,

$$\frac{d\rho}{dt} + \frac{d.\rho u}{dx} + \frac{d.\rho v}{dy} + \frac{d.\rho w}{dz} = 0 \dots\dots\dots(1)$$

$$(dp) = \rho\left\{\left[X - \left(\frac{du}{dt}\right)\right]dx + \left[Y - \left(\frac{dv}{dt}\right)\right]dy + \left[Z - \left(\frac{dw}{dt}\right)\right]dz\right\} \dots(2).$$

And we have also,

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt}.$$

It will be proper to explain here, that in the above equations, and in the subsequent investigation, the following notation is adopted for the sake of perspicuity. The *differential coefficients* of the quantities p, ρ, u, v, w, are *partial* when they are not in brackets; when in brackets they are *complete*, the variation being with respect both to the time and the three co-ordinates. A *differential*, as (dp), is put in brackets to indicate that the variation is with respect to the *three co-ordinates*, the time being given.

By substituting P for a^2 Nap. log ρ, and (dQ) for $Xdx + Ydy + Zdz$, regarding X, Y, Z, as functions both of t and the co-ordinates, the equation (2) will be changed to the following:

$$(dP) - (dQ) + \left(\frac{du}{dt}\right) dx + \left(\frac{dv}{dt}\right) dy + \left(\frac{dw}{dt}\right) dz = 0 \ldots\ldots\ldots\ldots\ldots (3).$$

2. That the above equations may be available for application to proposed instances of motion, it is required to derive from them a single partial differential equation in which the principal variable is a function of x, y, z, and t. This has been long done on the particular hypothesis that $udx + vdy + wdz$ is integrable *per se.* I propose to give some consideration to this case, preparatory to the more general investigation that will follow.

On the above hypothesis we may assume ϕ to be a function of x, y, z and t, such that,

$$(d\phi) = udx + vdy + wdz.$$

Consequently,

$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad w = \frac{d\phi}{dz}.$$

3. If the fluid be incompressible ρ is constant, $\dfrac{d\rho}{dt} = 0$, and equation (1) becomes,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

Hence for this case the required partial differential equation is evidently,

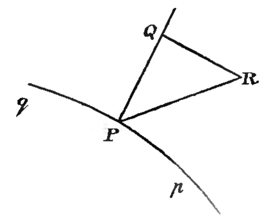
$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0.$$

I proceed to make a transformation of this equation which will be serviceable in the subsequent calculations.

First, it will appear by the following reasoning that

$$udx + vdy + wdz = 0,$$

is the differential equation of a surface which at a given instance cuts at right angle the directions of motion of the particles through which it passes*. Let qPp (in the figure) represent such a surface, which, for brevity, we will call a surface of *displacement*. Let P be a point in it, the co-ordinates of which are x, y, z, and let R be any other point indefinitely near, whose co-ordinates are $x + dx$, $y + dy$, $z + dz$. Draw PQ in the direction of the motion at P, and therefore perpendicular to the surface qPp, and draw RQ perpendicular to PQ. Let $PR = ds$, and $PQ = dr$. Also let PR make the angles α, β, γ with the axes of co-ordinates, PQ make the angles α', β', γ' with the same axes, and PR make the angle θ with PQ. Then if V be the velocity at P,



$$u = V\cos\alpha', \quad v = V\cos\beta', \quad w = V\cos\gamma',$$

$$\text{also,} \quad dx = ds\cos\alpha, \quad dy = ds\cos\beta, \quad dz = ds\cos\gamma.$$

Hence,

$$udx + vdy + wdz = Vds\,(\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma')$$
$$= Vds\cos\theta = Vdr.$$

Now if the variation of the co-ordinates be from P to a point p indefinitely near on the surface of displacement, $dr = 0$, and therefore, since V does not vanish,

$$udx + vdy + wdz = 0,$$

which it was required to prove.

Next, if r and r' be the principal radii of curvature at any point of a surface, the differential equation of which is $(d\phi) = 0$, it may be shewn by the processes of Analytical Geometry, that,

* See Mr. Earnshaw on Fluid Motion in the Cambridge Philosophical Transactions, Vol. VI. Part II. p. 204.

$$\frac{1}{r} + \frac{1}{r'} = \left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right)^{-\frac{3}{2}}.$$

$$\left\{\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right)\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right) - \frac{d^2\phi}{dx^2}\cdot\frac{d\phi^2}{dx^2} - \frac{d^2\phi}{dy^2}\cdot\frac{d\phi^2}{dy^2} - \frac{d^2\phi}{dz^2}\cdot\frac{d\phi^2}{dz^2}\right.$$

$$\left. - 2\cdot\frac{d^2\phi}{dx\,dy}\cdot\frac{d\phi}{dx}\cdot\frac{d\phi}{dy} - 2\cdot\frac{d^2\phi}{dx\,dz}\cdot\frac{d\phi}{dx}\cdot\frac{d\phi}{dz} - 2\cdot\frac{d^2\phi}{dy\,dz}\cdot\frac{d\phi}{dy}\cdot\frac{d\phi}{dz}\right\}.$$

Now $(d^2\phi) = \frac{d^2\phi}{dx^2}\cdot dx^2 + \frac{d^2\phi}{dy^2}dy^2 + \frac{d^2\phi}{dz^2}dz^2$

$$+ 2\frac{d^2\phi}{dx\,dy}\cdot dx\,dy + 2\cdot\frac{d^2\phi}{dx\,dz}dx\,dz + 2\cdot\frac{d^2\phi}{dy\,dz}dy\,dz;$$

and if we assume the variation in $(d^2\phi)$ to be from one point to another in *the line of motion*, we shall have

$$dx = u\,dt = \frac{d\phi}{dx}dt,$$

$$dy = v\,dt = \frac{d\phi}{dy}dt,$$

$$dz = w\,dt = \frac{d\phi}{dz}dt.$$

Hence, by substituting these values of dx, dy, dz in the expression for $(d^2\phi)$ we obtain,

$$\frac{(d^2\phi)}{dt^2} = \frac{d^2\phi}{dx^2}\cdot\frac{d\phi^2}{dx^2} + \frac{d^2\phi}{dy^2}\cdot\frac{d\phi^2}{dy^2} + \frac{d^2\phi}{dz^2}\cdot\frac{d\phi^2}{dz^2}$$

$$+ 2\cdot\frac{d^2\phi}{dx\,dy}\cdot\frac{d\phi}{dx}\cdot\frac{d\phi}{dy} + 2\cdot\frac{d^2\phi}{dx\,dz}\cdot\frac{d\phi}{dx}\cdot\frac{d\phi}{dz} + 2\cdot\frac{d^2\phi}{dy\,dz}\cdot\frac{d\phi}{dy}\cdot\frac{d\phi}{dz}.$$

Now if s be a line drawn at a given instant in the direction of the motion of the particles through which it passes, and V be the velocity at the point xyz of this line at the time t, $V = \frac{ds}{dt}$, or $dt = \frac{ds}{V}$.

Hence, $\frac{(d^2\phi)}{dt^2} = V^2\cdot\frac{(d^2\phi)}{ds^2}$. But $\frac{(d\phi)}{ds} = u\cdot\frac{dx}{ds} + v\cdot\frac{dy}{ds} + w\cdot\frac{dz}{ds}$;

and the variations dx, dy, dz, being supposed to take place from one point to another in the line of motion,

$$\frac{dx}{ds} = \frac{u}{V}, \quad \frac{dy}{ds} = \frac{v}{V}, \quad \frac{dz}{ds} = \frac{w}{V};$$

so that, $\frac{(d\phi)}{ds} = \frac{1}{V}(u^2 + v^2 + w^2) = V,$

and $\frac{(d^2\phi)}{ds^2} = \frac{dV}{ds}$. Also $V^2 = \frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}$.

Hence by substituting in the foregoing value of $\frac{1}{r} + \frac{1}{r'}$, we obtain,

$$\frac{1}{r} + \frac{1}{r'} = \frac{1}{V^3}\left\{\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right) V^2 - V^2\frac{dV}{ds}\right\};$$

$$\text{or,}\ \frac{dV}{ds} + V\left(\frac{1}{r} + \frac{1}{r'}\right) = \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}.$$

The transformed equation sought for is therefore,

$$\frac{dV}{ds} + V\left(\frac{1}{r} + \frac{1}{r'}\right) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

4. A similar transformation of equation (1) may readily be effected when ρ is variable. For this equation may be put under the form,

$$\frac{d\rho}{\rho dt} + \frac{d\rho}{\rho dx}u + \frac{d\rho}{\rho dy}v + \frac{d\rho}{\rho dz}w + \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

Hence putting $\frac{dx}{dt}$ for u, $\frac{dy}{dt}$ for v, and $\frac{dz}{dt}$ for w, we have

$$\frac{d\rho}{\rho dt} + \frac{1}{a^2}\cdot\frac{(dP)}{dt} + \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

But it must be observed that the variation in (dP) is from one point to another *in the line of motion*, on account of the above substitutions for u, v, and w.

Hence $\frac{(dP)}{dt} = V.\frac{(dP)}{ds} = Va^2.\frac{d\rho}{\rho ds}$. Also, as has already been proved,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = \frac{dV}{ds} + V\left(\frac{1}{r} + \frac{1}{r'}\right).$$

Therefore by substitution,

$$\frac{d\rho}{dt} + \frac{d.V\rho}{ds} + V\rho\left(\frac{1}{r} + \frac{1}{r'}\right) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

I have obtained this equation in an entirely different manner in the Transactions of the Cambridge Philosophical Society, Vol. V. Part II, p. 196.

5. The equation (3), by substituting $\frac{d\phi}{dx}$ for u, $\frac{d\phi}{dy}$ for v, and $\frac{d\phi}{dz}$ for w, is transformed into another, which by integration gives,

$$P - Q + \frac{d\phi}{dt} + \frac{1}{2}\cdot\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right) = F(t) \ldots\ldots\ldots\ldots(6).$$

Now since $V = \frac{d\phi}{ds}$, $\frac{dV}{dt} = \frac{d^2\phi}{dsdt}$. Hence $\frac{d\phi}{dt} = \int\frac{dV}{dt}ds$, the integration being performed along the line s. The above equation thus becomes,

$$P - Q + \int\frac{dV}{dt}ds + \frac{V^2}{2} = F(t) \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Hence, by differentiating with respect to t,

$$\frac{dP}{dt}, \text{ or } \frac{a^2 d\rho}{\rho dt} = \frac{dQ}{dt} - \int\frac{d^2V}{dt^2}ds - V\frac{dV}{dt} + F'(t);$$

and by differentiating with respect to s,

$$\frac{dP}{ds}, \text{ or } \frac{a^2 d\rho}{\rho ds} = \frac{dQ}{ds} - \frac{dV}{dt} - V.\frac{dV}{ds}.$$

Consequently by substituting in equation (5) we obtain,

$$F'(t) + \frac{dQ}{dt} + V.\frac{dQ}{ds} + (a^2 - V^2)\frac{dV}{ds} - \int\frac{d^2V}{dt^2}ds - 2V.\frac{dV}{dt} + Va^2\left(\frac{1}{r} + \frac{1}{r'}\right) = 0 \ldots(8).$$

If now $\frac{d\phi}{ds}$ be substituted for V, we have $\int\frac{d^2V}{dt^2}ds = \int\frac{d^3\phi}{dsdt^2}ds = \frac{d^2\phi}{dt^2}$, and the above equation becomes,

$$F'(t)+\frac{dQ}{dt}+\frac{d\phi}{ds}\cdot\frac{dQ}{ds}+\left(a^2-\frac{d\phi^2}{ds^2}\right)\frac{d^2\phi}{ds^2}-\frac{d^2\phi}{dt^2}-2\cdot\frac{d\phi}{ds}\cdot\frac{d^2\phi}{dsdt}+a^2\cdot\frac{d\phi}{ds}\cdot\left(\frac{1}{r}+\frac{1}{r'}\right)=0\ldots(9).$$

The arbitrary function of the time $F'(t)$ may be got rid of by substituting $\phi' + \int F(t)dt$ for ϕ.

Again, by differentiating equation (8) with respect to s, an equation of the same order as the preceding may be obtained, having V for principal variable. The result (since $dr = dr' = ds$) will be,

$$\frac{d^2Q}{ds\,dt}+\frac{dV}{ds}\cdot\frac{dQ}{ds}+V\cdot\frac{d^2Q}{ds^2}+(a^2-V^2)\frac{d^2V}{ds^2}-\frac{d^2V}{dt^2}-2V\cdot\frac{dV^2}{ds^2}$$

$$-2V\cdot\frac{d^2V}{ds\,dt}-2\cdot\frac{dV}{ds}\cdot\frac{dV}{dt}+a^2\cdot\frac{dV}{ds}\left(\frac{1}{r}+\frac{1}{r'}\right)-Va^2\left(\frac{1}{r^2}+\frac{1}{r'^2}\right)=0\ldots\ldots(10).$$

It should be observed that equations (9) and (10) are subject to the same limitation as equation (5) in regard to the direction of the variation of the co-ordinates.

6. All the preceding results have been obtained on the supposition that $udx + vdy + wdz$ is an exact differential. It will now be proper to enquire to what circumstances of the motion this analytical condition refers. The following considerations will enable us to do this.

It has already been proved in Art. 3 that,

$$udx + vdy + wdz = Vdr;$$

in which dr may be considered the increment of a straight line drawn in the direction of the motion, and dx, dy, dz are the corresponding increments of the co-ordinates, the variations taking place at a given instant from one point to another indefinitely near. Now Vdr is not an exact differential unless V may be considered a function of the line r; that is, unless the variation of V from one point of space to another at a given instant depends only on the change of position in the direction normal to the surface of displacement, the variation from one point to another of the surface of displacement being zero. Therefore also $udx + vdy + wdz$ is not an exact differential unless $dV = 0$ when the co-ordinates vary at a given instant from one point to another of a given surface of displacement.*

* See a direct proof of this Proposition in the 'Note' added to this Paper.

If now equation (6) be differentiated with respect to space, the result is,

$$dP - dQ + d.\frac{d\phi}{dt} + VdV = 0;$$

and as this equation is subject to no limitation with respect to the direction of variation of co-ordinates, let us suppose the variation to be from one point to another of a surface of displacement. Then from what has been just shewn, $dV = 0$. Also

$$d.\frac{d\phi}{dt} = \frac{d^2\phi}{dxdt}dx + \frac{d^2\phi}{dydt}dy + \frac{d^2\phi}{dzdt}dz = \frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz$$

$$= \frac{d.(udx + vdy + wdz)}{dt} = 0,$$

because for a surface of displacement $udx + vdy + wdz = 0$. Consequently $dP - dQ = 0$. It follows from this that when $udx + vdy + wdz$ is an exact differential, the surface of displacement coincides with a surface for all points of which $P - Q$ has the same value.

If $Q = 0$, that is, if there be no impressed forces, the surface of displacement evidently coincides with a surface of equal pressure, and the motion of each fluid particle must consequently be *rectilinear*. In this case only equation (5) is subject to no limitation in regard to the direction of the variation of the co-ordinates.

The motion is rectilinear also when Q is not equal to nothing. For though in this case the pressure varies along a surface of displacement, the effect of this variation is just counterbalanced by the impressed forces, as may be thus shewn. Let $d\sigma$ be the increment of any line drawn on the surface of displacement. Then

$$-\frac{dP}{d\sigma} + \frac{dQ}{d\sigma} = -\frac{a^2 d\rho}{\rho d\sigma} + X\frac{dx}{d\sigma} + Y\frac{dy}{d\sigma} + Z\frac{dz}{d\sigma} = \frac{d^2\sigma}{dt^2},$$

the effective accelerative force in the direction of σ. Since therefore,

$$-\frac{dP}{d\sigma} + \frac{dQ}{d\sigma} = 0,$$

the effective accelerative force in any direction along the surface of displacement is nothing; and the velocity being the same at all points of this surface, it follows that the motion is rectilinear.

We may therefore conclude that *in the instances of fluid motion for which* u dx + v dy + w dz *is an exact differential of a function of three independent variables, the motion of every particle of the fluid is rectilinear.*

7. Hence in equations (9) and (10) the radii of curvature r, r', pass through fixed points or fixed focal lines, and the line s coincides with r. Hence changing ds into dr in equation (9), substituting $\phi' + \int F(t)dt$ for ϕ, and suppressing the accent, we obtain,

$$\frac{dQ}{dt} + \frac{d\phi}{dr} \cdot \frac{dQ}{dr} + \left(a^2 - \frac{d\phi^2}{dr^2}\right)\frac{d^2\phi}{dr^2} - \frac{d^2\phi}{dt^2} - 2\frac{d\phi}{dr} \cdot \frac{d^2\phi}{drdt} + a^2 \cdot \frac{d\phi}{dr}\left(\frac{1}{r} + \frac{1}{r'}\right) = 0 \ldots (11)$$

which equation is applicable to any instance whatever of rectilinear fluid motion.

8. I proceed now to the consideration of the more general case, viz. that in which $udx + vdy + wdz$ becomes integrable by being multiplied by a factor*. Let $\frac{1}{N}$ be the factor. Then we may assume the function ϕ to be such that,

$$(d\phi) = \frac{u}{N}dx + \frac{v}{N}dy + \frac{w}{N}dz,$$

so that we have,

$$u = N.\frac{d\phi}{dx}, \qquad v = N.\frac{d\phi}{dy}, \qquad w = N.\frac{d\phi}{dz}.$$

The introduction of this new quantity N makes an additional equation necessary by which it may be determined. This may be investigated as follows. By the reasoning of Art. 3, it appears that

$$\frac{u}{N}dx + \frac{v}{N}dy + \frac{w}{N}dz = \frac{V}{N}dr;$$

and if the variation be from one point to another of a surface of displacement $dr = 0$. Hence the equation,

$$\frac{u}{N}dx + \frac{v}{N}dy + \frac{w}{N}dz = 0,$$

* Mr. Earnshaw has suggested the idea of multiplying by a factor, in the Paper on Fluid Motion already referred to.

being by hypothesis integrable, is the differential equation of a surface of displacement. The integral of this equation, since the left-hand side of it is equal to $(d\phi)$, is $\phi = 0$, an arbitrary function of the time being included in ϕ. As this reasoning applies to the whole fluid during the whole time of its motion, there will at each instant be an unlimited number of surfaces of displacement differing according to different values assigned to the arbitrary parameters involved in ϕ; also, as ϕ contains the time t in any arbitrary manner, these surfaces may be supposed to be continually changing their forms and positions. Consequently if x, y, z, be the co-ordinates of a given surface of displacement at the time t, $x + udt$, $y + vdt$, $z + wdt$, will be the co-ordinates of the same surface in the form and position which it takes at the time $t + dt$, the change of form and position being supposed to be indefinitely small. If therefore t be changed to $t + dt$, and x, y, z, be changed to $x + udt$, $y + vdt$, $z + wdt$, in the equation $\phi = 0$, that equation will still be satisfied. Hence,

$$\phi\,(x + udt, \quad y + vdt, \quad z + wdt, \quad t + dt) = 0,$$

and expanding to the first powers of dt,

$$\phi + \frac{d\phi}{dx}\,udt + \frac{d\phi}{dy}\,vdt + \frac{d\phi}{dz}\,wdt + \frac{d\phi}{dt}\,dt = 0,$$

which equation, since $\phi = 0$, becomes,

$$\frac{d\phi}{dt} + \frac{d\phi}{dx}\,u + \frac{d\phi}{dy}\,v + \frac{d\phi}{dz}\,w = 0 \ldots\ldots\ldots\ldots\ldots (12).$$

Now substituting $N\frac{d\phi}{dx}$ for u, $N\frac{d\phi}{dy}$ for v, and $N\frac{d\phi}{dz}$ for w, we obtain the equation sought, viz.

$$\frac{d\phi}{dt} + N\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right) = 0 \ldots\ldots\ldots\ldots\ldots (13).$$

9. If in equation (12) $\frac{dx}{dt}$ be substituted for u, $\frac{dy}{dt}$ for v, and $\frac{dz}{dt}$ for w, the result may be put under the form $\frac{d\phi}{dt} + \frac{(d\phi)}{dt} = 0$. But it must be

borne in mind that on account of these substitutions, the variation in $(d\phi)$ is from one point to another *in the line of motion.*

10. Resuming the fundamental equation (3), and substituting in it the values of u, v, w, we have

$$(dP) - (dQ) + \left(\frac{d\,.\,N\frac{d\phi}{dx}}{dt}\right)dx + \left(\frac{d\,.\,N\frac{d\phi}{dy}}{dt}\right)dy + \left(\frac{d\,.\,N\frac{d\phi}{dz}}{dt}\right)dz = 0.$$

$$\text{Now } \left(\frac{d\,.\,\frac{d\phi}{dx}}{dt}\right) = \frac{d^2\phi}{dx\,dt} + \frac{d^2\phi}{dx^2}\cdot\frac{dx}{dt} + \frac{d^2\phi}{dx\,dy}\cdot\frac{dy}{dt} + \frac{d^2\phi}{dx\,dz}\cdot\frac{dz}{dt}$$

$$= \frac{d^2\phi}{dx\,dt} + N\left(\frac{d\phi}{dx}\cdot\frac{d^2\phi}{dx^2} + \frac{d\phi}{dy}\cdot\frac{d^2\phi}{dx\,dy} + \frac{d\phi}{dz}\cdot\frac{d^2\phi}{dx\,dz}\right);$$

and similarly for $\left(\frac{d\,.\,\frac{d\phi}{dy}}{dt}\right)$ and $\left(\frac{d\,.\,\frac{d\phi}{dz}}{dt}\right)$.

Hence by performing the differentiations of the foregoing equation, the result expressed in the notation already used, will be

$$(dP) - (dQ) + N\,.\left(d\,.\frac{d\phi}{dt}\right) + \frac{N^2}{2}\,.\left(d\,.\left\{\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right\}\right)$$
$$+ \left\{\frac{dN}{dt} + \frac{(dN)}{dt}\right\}(d\phi) = 0.$$

Therefore by integration,

$$F(t) + P - Q + N\frac{d\phi}{dt} + \int\left\{\frac{dN}{dt} + \frac{(dN)}{dt}\right\}(d\phi)$$
$$+ \frac{N^2}{2}\cdot\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right) + \int\left\{\frac{d\phi}{dt} + N\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right)\right\}(dN) = 0.$$

The equation (13), makes the last term disappear.

11. If N be a function of t only,

$$\frac{(dN)}{dt} = 0, \quad \text{and} \int\frac{dN}{dt}(d\phi) = \phi\,\frac{dN}{dt}.$$

Hence, if we substitute ψ for $N\phi$, the preceding equation will become,

$$F(t) + P - Q + \frac{d\psi}{dt} + \tfrac{1}{2} \cdot \left(\frac{d\psi^2}{dx^2} + \frac{d\psi^2}{dy^2} + \frac{d\psi^2}{dz^2}\right) = 0,$$

which is the equation obtained when $udx + vdy + wdz$ is an exact differential, as it manifestly ought to be. The general expression for ϕ inclusive of all the cases in which that differential is exact, would be obtained by integrating the equation,

$$\frac{d\phi}{dt} + N\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right) = 0,$$

on the supposition that N is a function of t only. By multiplying this equation by N, it becomes $N\frac{d\phi}{dt} + V^2 = 0$; whence it appears that since $\frac{d\phi}{dt}$ and V are constant for a given surface of displacement at a given time when $udx + vdy + wdz$ is an exact differential (see Art. 6), the factor N is constant under the same circumstances. With this limitation, therefore, as to the value of N, the equation (13), holds good at the same time that $udx + vdy + wdz$ is integrable of itself.

12. Resuming the equation obtained in Art. 10, we have in general,

$$F(t) + P - Q + N\frac{d\phi}{dt} + \int\left\{\frac{dN}{dt} + \frac{(dN)}{dt}\right\}(d\phi) + \frac{V^2}{2} = 0.$$

But $\left\{\frac{dN}{dt} + \frac{(dN)}{dt}\right\}(d\phi) = \frac{dN}{dt}(d\phi) + (dN)\frac{(d\phi)}{dt} = \frac{dN}{dt}(d\phi) - (dN)\frac{d\phi}{dt}$

by Art. 9, if the variation with respect to space be from one point to another in the line of motion. Also,

$$\int\left\{\frac{dN}{dt}(d\phi) - (dN)\frac{d\phi}{dt}\right\} = -N\frac{d\phi}{dt} + \int\left\{\frac{dN}{dt}(d\phi) + N\left(d.\frac{d\phi}{dt}\right)\right\}$$

$$= -N\frac{d\phi}{dt} + \int\left\{\frac{dN}{dt}(d\phi) + N.\frac{d(d\phi)}{dt}\right\}$$

$$= -N\frac{d\phi}{dt} + \frac{d.\int N(d\phi)}{dt}.$$

Hence by substitution,

$$F(t) + P - Q + \frac{d . \int N(d\phi)}{dt} + \frac{V^2}{2} = 0.$$

Again, on the above limitation respecting the variation from one point to another of space,

$$N(d\phi) = N\left(\frac{d\phi}{dx} \cdot \frac{dx}{ds} + \frac{d\phi}{dy} \cdot \frac{dy}{ds} + \frac{d\phi}{dz} \cdot \frac{dz}{ds}\right) ds = \left(u . \frac{u}{V} + v . \frac{v}{V} + w . \frac{w}{V}\right) ds = V ds.$$

Hence $$\frac{d \int N(d\phi)}{dt} = \int \frac{dV}{dt} ds .$$

The foregoing equation consequently becomes,

$$F(t) + P - Q + \int \frac{dV}{dt} ds + \frac{V^2}{2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

which is precisely the same in form as equation (7), and differs only in being limited as to the direction of variation of the co-ordinates. The same equation subject to the same limitation may be obtained directly from equation (2) as follows.

If F be the sum of the impressed forces and f the effective accelerative force in the direction of an arbitrary line s drawn in the mass of fluid in motion, then by D'Alembert's Principle and Hydrostatics,

$$dp = \rho (F - f) ds.$$

But $F ds = \left(X . \frac{dx}{ds} + Y . \frac{dy}{ds} + Z . \frac{dz}{ds}\right) ds = X dx + Y dy + Z dz.$

Hence, by integration,

$$P - Q + \int f ds = F(t).$$

If V be the velocity in the direction of s,

$$f = \left(\frac{dV}{dt}\right) = \frac{dV}{dt} + \frac{(dV)}{dt},$$

indicating by (dV) as before,

$$\frac{dV}{dx} dx + \frac{dV}{dy} dy + \frac{dV}{dz} dz .$$

Now if V be the *whole* velocity and not otherwise, that is, if the line s be drawn in the direction of the motion,

$$\frac{dx}{dt} = V.\frac{dx}{ds}, \qquad \frac{dy}{dt} = V.\frac{dy}{ds}, \qquad \frac{dz}{dt} = V.\frac{dz}{ds},$$

and $$\frac{(dV)}{dt} = V\left(\frac{dV}{dx}.\frac{dx}{ds} + \frac{dV}{dy}.\frac{dy}{ds} + \frac{dV}{dz}.\frac{dz}{ds}\right) = V.\frac{dV}{ds}.$$

Hence by substituting the resulting value of f in the above equation,

$$P - Q + \int\frac{dV}{dt}ds + \frac{V^2}{2} = F(t).$$

In consequence of the limitation to which this equation is subject, it cannot be argued, as in the former case, that the motion is rectilinear, and we may therefore conclude that *when* u dx + v dy + w dz *is integrable by a factor the motion is curvilinear.*

Another remark may also be made here. By assigning a given value to $P - Q$, the above equation becomes the equation of a surface for all points of which $P - Q$ has that value at a given instant. Hence differentiating the equation with respect to space and putting the differential under the form

$$dP - dQ = -\frac{dV}{dt}ds - VdV,$$

the variation of $P - Q$ on the left-hand side of the equation will be the same in passing from a point of that surface to a point of another such surface indefinitely near, whatever be the relative position of the points. We may therefore suppose the variation to take place from one to the other of the points of intersection of the line of motion with these two surfaces, to which variation the quantities on the right-hand side are limited. Hence the equation holds good notwithstanding that limitation.

13. Our next step will be to effect a transformation of equation (1) analogous to that which was made in Art. 4 on the supposition that $udx + vdy + wdz$ is an exact differential. The transformation will now be made by means of equation (13), and therefore on the supposition that that quantity is integrable by a factor $\frac{1}{N}$.

Since $u = N\frac{d\phi}{dx}$, by equation (13),

$$\frac{d\phi}{dx}\cdot\frac{d\phi}{dt} + u\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right) = 0;$$

whence by differentiating with respect to x,

$$\frac{du}{dx} = -\frac{\frac{d^2\phi}{dx^2}\cdot\frac{d\phi}{dt} + \frac{d^2\phi}{dxdt}\cdot\frac{d\phi}{dx}}{\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}} + \frac{2\frac{d\phi}{dt}\cdot\frac{d\phi}{dx}\left(\frac{d\phi}{dx}\cdot\frac{d^2\phi}{dx^2} + \frac{d\phi}{dy}\cdot\frac{d^2\phi}{dxdy} + \frac{d\phi}{dz}\cdot\frac{d^2\phi}{dxdz}\right)}{\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right)^2}.$$

Similar expressions having been obtained for $\frac{dv}{dy}$ and $\frac{dw}{dz}$, it will be found by adding them together, and having regard to the formula in Art. 2, for $\frac{1}{r} + \frac{1}{r'}$, that,

$$\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right) = \frac{d^2\phi}{dx^2}\cdot\frac{d\phi}{dt} - \frac{d^2\phi}{dxdt}\cdot\frac{d\phi}{dx} + \frac{d^2\phi}{dy^2}\cdot\frac{d\phi}{dt} - \frac{d^2\phi}{dydt}\cdot\frac{d\phi}{dy}$$

$$+ \frac{d^2\phi}{dz^2}\cdot\frac{d\phi}{dt} - \frac{d^2\phi}{dzdt}\cdot\frac{d\phi}{dz} + 2\frac{d\phi}{dt}\cdot\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right)^{\frac{1}{2}}\left(\frac{1}{r} + \frac{1}{r'}\right),$$

which equation may be reduced as follows to one of a simpler form. Equation (13) gives,

$$\frac{d\phi^2}{dt^2} = N^2\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right)^2 = V^2\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right).$$

Hence, $$\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2} = \frac{1}{V^2}\cdot\frac{d\phi^2}{dt^2}.$$

Also, by multiplying equation (13) by $N\frac{d\phi}{dx}$, it will appear that,

$$u\frac{d\phi}{dt} + V^2\cdot\frac{d\phi}{dx} = 0;$$

and this equation by differentiating with respect to x gives,

$$\frac{d^2\phi}{dx^2}\cdot\frac{d\phi}{dt}-\frac{d^2\phi}{dxdt}\cdot\frac{d\phi}{dx}=\frac{1}{V^2}\cdot\frac{d\phi^2}{dt^2}\left(\frac{2u}{V}\cdot\frac{dV}{dx}-\frac{du}{dx}\right).$$

So $$\frac{d^2\phi}{dy^2}\cdot\frac{d\phi}{dt}-\frac{d^2\phi}{dydt}\cdot\frac{d\phi}{dy}=\frac{1}{V^2}\cdot\frac{d\phi^2}{dt^2}\left(\frac{2v}{V}\cdot\frac{dV}{dy}-\frac{dv}{dy}\right).$$

And $$\frac{d^2\phi}{dz^2}\cdot\frac{d\phi}{dt}-\frac{d^2\phi}{dzdt}\cdot\frac{d\phi}{dz}=\frac{1}{V^2}\cdot\frac{d\phi^2}{dt^2}\left(\frac{2w}{V}\cdot\frac{dV}{dz}-\frac{dw}{dz}\right).$$

When the several values thus obtained are substituted in the foregoing equation, the result is,

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=\frac{u}{V}\cdot\frac{dV}{dx}+\frac{v}{V}\frac{dV}{dy}+\frac{w}{V}\cdot\frac{dV}{dz}+V\left(\frac{1}{r}+\frac{1}{r'}\right).$$

If now the condition be introduced that the variation from one point to another of space be in the line of motion, we shall have,

$$\frac{u}{V}=\frac{dx}{ds},\qquad \frac{v}{V}=\frac{dy}{ds},\qquad \frac{w}{V}=\frac{dz}{ds};$$

and the above result is reduced to the following,

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=\frac{dV}{ds}+V\left(\frac{1}{r}+\frac{1}{r'}\right).$$

Consequently by reasoning exactly as in Art. 4, an equation the same as (5) results: and by eliminating ρ from this equation by means of equation (14), the equation (10) is reproduced. We may therefore conclude that *the same differential equation of the second order, in which* V *is the principal variable, applies to curvilinear as to rectilinear motion, the variation of the co-ordinates at a given time being from one point to another in the line of motion.*

14. The reason of this result will be seen by the following considerations. Conceive two surfaces of displacement to be drawn at a given instant indefinitely near each other, one of which passes through the point P given in position. On this surface describe an indefinitely small rectangular area having P at its centre, and having its sides in planes of greatest and least curvature. On the other surface take a similar

area, such in magnitude and position that the straight lines joining the corresponding angular points of the two areas are normals to the first surface. By the nature of curve surfaces these normals will meet two and two in two focal lines situated in the planes of greatest and least curvature, and cutting the normals at right angles. Let the small area of which P is the centre be m^2, and let r, r' be the distances of the focal lines from P. Then if the positions of the focal lines do not vary with the time, the other area is ultimately $\frac{m^2(r+\delta r).(r'+\delta r)}{rr'}$, the interval between the two surfaces of displacement being a given small quantity δr. This is the case of *rectilinear* motion. But if the direction of the motion through P is continually changing, the surface of displacement through that point will vary with the time. Hence the positions of the focal lines and the magnitudes of r and r' will change continually, whilst the area m^2 may be supposed to remain the same and always to pass through the point P. Let r and r' represent the values of the principal radii of curvature at the time t, and let α and β be the velocities of the focal lines estimated in the direction of the radii of curvature and considered positive when the motion is *towards* P. Then at the time $t+\delta t$ the values of r and r' become $r-\alpha\delta t$ and $r'-\beta\delta t$, and the elementary area on the second surface is

$$m^2 . \frac{(r+\delta r-\alpha\delta t)(r'+\delta r-\beta\delta t)}{(r-\alpha\delta t)(r'-\beta\delta t)},$$

which is equal to

$$m^2 . \frac{(r+\delta r)(r'+\delta r)}{rr'} \times \frac{\left(1-\frac{\alpha\delta t}{r+\delta r}\right)\left(1-\frac{\beta\delta t}{r'+\delta r}\right)}{\left(1-\frac{\alpha\delta t}{r}\right)\left(1-\frac{\beta\delta t}{r'}\right)},$$

$$\text{or, } m^2 . \frac{(r+\delta r)(r'+\delta r)}{rr'} \times \left(1+\frac{\alpha\delta t\,\delta r}{r^2}\right)\left(1+\frac{\beta\delta t\,\delta r}{r'^2}\right) \text{ ultimately.}$$

Hence, by omitting quantities of the order of $\frac{\alpha\delta t}{r} \times \frac{\delta r}{r}$, the result is the same as when the position of the focal lines is supposed to be fixed. If therefore V and ρ be the velocity and density of the fluid which passes the area m^2, and V' and ρ' the velocity and density of the fluid which simultaneously passes the other area, then assuming these quantities (as is permitted) to be uniform during the small time δt, and considering the velocity positive when directed *from* the focal lines, the increment of matter between the two areas in the time δt is

$$-m^2 . \frac{(r+\delta r)(r'+\delta r)}{rr'} \rho' V' \delta t + m^2 . \rho V \delta t,$$

$$\text{or,} \quad - m^2 \delta t \left\{ \frac{d . \rho V}{dr} + \rho V \left(\frac{1}{r} + \frac{1}{r'} \right) \right\} \delta r \quad \text{ultimately.}$$

And this quantity is also equal to $m^2 \delta\rho \delta r$ ultimately. Hence

$$m^2 \delta\rho \delta r + m^2 \delta t \delta r \left\{ \frac{d . \rho V}{dr} + \rho V \left(\frac{1}{r} + \frac{1}{r'} \right) \right\} = 0.$$

And passing from differences to differentials,

$$\frac{d\rho}{dt} + \frac{d . \rho V}{dr} + \rho V \left(\frac{1}{r} + \frac{1}{r'} \right) = 0,$$

which coincides with equation (5).

15. To complete our investigation it will now be requisite to obtain the partial differential equation containing the variables ϕ, x, y, z and t, ϕ being the principal variable. This is readily done in the case of an incompressible fluid. For substituting $N \frac{d\phi}{dx}$ for u, $N \frac{d\phi}{dy}$ for v, and $N \frac{d\phi}{dz}$ for w, in the equation $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$, the result is,

$$\frac{dN}{dx} . \frac{d\phi}{dx} + \frac{dN}{dy} . \frac{d\phi}{dy} + \frac{dN}{dz} . \frac{d\phi}{dz} + N \left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} \right) = 0.$$

And eliminating N by means of the equation

$$\frac{d\phi}{dt} + N \left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2} \right) = 0,$$

the required equation is found to be,

$$\left(\frac{d\phi^2}{dx^2}+\frac{d\phi^2}{dy^2}+\frac{d\phi^2}{dz^2}\right)\left\{\frac{d\phi}{dt}\cdot\frac{d^2\phi}{dx^2}+\frac{d\phi}{dx}\cdot\frac{d^2\phi}{dxdt}+\frac{d\phi}{dt}\cdot\frac{d^2\phi}{dy^2}+\frac{d\phi}{dy}\cdot\frac{d^2\phi}{dydt}+\frac{d\phi}{dt}\cdot\frac{d^2\phi}{dz^2}+\frac{d\phi}{dz}\cdot\frac{d^2\phi}{dzdt}\right\}$$

$$-2\frac{d\phi}{dt}\cdot\left\{\frac{d\phi^2}{dx^2}\cdot\frac{d^2\phi}{dx^2}+\frac{d\phi^2}{dy^2}\cdot\frac{d^2\phi}{dy^2}+\frac{d\phi^2}{dz^2}\cdot\frac{d^2\phi}{dz^2}\right.$$

$$\left.+2\frac{d\phi}{dx}\cdot\frac{d\phi}{dy}\cdot\frac{d^2\phi}{dxdy}+2\cdot\frac{d\phi}{dx}\cdot\frac{d\phi}{dz}\cdot\frac{d^2\phi}{dxdz}+2\frac{d\phi}{dy}\cdot\frac{d\phi}{dz}\cdot\frac{d^2\phi}{dydz}\right\}=0.$$

According to the views maintained in this Essay, the above is the equation that should be employed in the Theory of the Tides: but it is probably too complicated to be available for that purpose. However, the simple equation,

$$\frac{dV}{ds}+V\left(\frac{1}{r}+\frac{1}{r'}\right)=0,$$

is integrable at once, and gives $V=\frac{\phi(t)}{rr'}$. And as, from what is shewn in Art. 14, the variation of V at a given point is the same as if r and r' were constant,

$\frac{dV}{dt}=\frac{\phi'(t)}{rr'}$. Hence if $r'=r+h$, $\int\frac{dV}{dt}dr=-\frac{\phi'(t)}{h}\text{ Nap. log. }\left(1+\frac{h}{r}\right)$.

Consequently by substituting in equation (14)

$$F(t)+P-Q-\frac{\phi'(t)}{h}\text{ Nap. log. }\left(1+\frac{h}{r}\right)+\frac{V^2}{2}=0.$$

It would be beside my purpose to inquire now into the applications that may be made of this equation.

16. A differential equation in which the principal variable ϕ is a function of x, y, z, and t, might also be obtained for the case of a compressible fluid, but it is of so complicated a nature that no use could be made of it, and I therefore omit writing it down. It is important to observe, that for curvilinear motions this equation rises to the *third* order. The inference to be drawn from this circumstance is, that the forms of the surfaces of displacement are entirely arbitrary in the general case of curvilinear motion, the three arbitrary functions which the complete integral contains, having to be determined by given conditions

respecting the velocity, the density, and the *form* of the surface of displacement. This consideration will enable us to draw some inferences in particular cases without having recourse to the general equation.

For example, the forms of the surfaces of displacement being any that we please, it may be assumed that a *given* surface of displacement, that is, one with which the *same* fluid particles remain in contact at successive instants, continues of a spherical form. Its differential equation will then be,

$$2x(x-\alpha)\,dx + 2y(y-\beta)\,dy + 2z(z-\gamma)\,dz = 0;$$

and the equation itself,

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = R^2;$$

in which α, β, γ, R, may either all be functions of the time, or part constant and part functions of the time. Let, for instance, α, β, γ be constant and R a function of the time. Then since

$$\phi = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - R^2,$$

it will be seen that

$$\frac{d\phi}{dx} = 2(x-\alpha), \quad \frac{d\phi}{dy} = 2(y-\beta), \quad \frac{d\phi}{dz} = 2(z-\gamma),$$

$$\text{and } \frac{d\phi}{dt} = -2R\frac{dR}{dt} = -2RV,$$

because R is by hypothesis the radius of the spherical surface in successive instants. Hence by substituting in the equation,

$$\frac{d\phi}{dt} + N.\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right) = 0,$$

the result is,

$$-2RV + 4NR^2 = 0; \quad \text{whence } N = \frac{V}{2R}.$$

Hence, as N is not a function of the co-ordinates x, y, z, the differential $udx + vdy + wdz$ is integrable of itself, which for this case it plainly should be, the motion being directed to or from a fixed centre.

17. Again, suppose α, β and R to be constant, and γ to vary with the time. This is to suppose the surface of displacement in successive instants to be that of a sphere of given radius moving in a direction parallel to the axis of z. We shall have

$$\frac{d\phi}{dt} = -2(z-\gamma)\frac{d\gamma}{dt}; \quad \frac{d\phi}{dx} = 2(x-\alpha), \quad \frac{d\phi}{dy} = 2(y-\beta), \quad \frac{d\phi}{dz} = 2(z-\gamma);$$

and therefore

$$\frac{d\phi}{dt} + N\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz}\right) = -2(z-\gamma)\frac{d\gamma}{dt} + 4NR^2 = 0.$$

$$\text{Hence } N = \frac{z-\gamma}{2R^2}\cdot\frac{d\gamma}{dt};$$

$$\text{and } V = N\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right)^{\frac{1}{2}} = \frac{z-\gamma}{2R^2}\cdot\frac{d\gamma}{dt}\cdot 2R = \frac{z-\gamma}{R}\cdot\frac{d\gamma}{dt}.$$

Now the general equation (10), when no impressed force is supposed to act, and when terms involving higher powers of V than the first are omitted, becomes for the case in which the motion is directed to or from a centre,

$$a^2\cdot\frac{d^2V}{dr^2} - \frac{d^2V}{dt} + \frac{2a^2}{r}\frac{dV}{dr} - \frac{2a^2V}{r^2} = 0;$$

the same equation being applicable whether the centre be moving or fixed, as is shewn in Art. 13. This equation is readily transformed into

$$\frac{d^2.Vr}{dt^2} = a^2\left(\frac{d^2.Vr}{dr^2} - \frac{2Vr}{r^2}\right),$$

the integral of which obtained by Euler (see Peacock's *Examples*, p. 473) gives,

$$V = -\frac{1}{r^2}\{f(r-at) + F(r+at)\} + \frac{1}{r}\{f'(r-at) + F'(r+at)\}.$$

If the arbitrary function F be supposed to vanish, the motion will be propagated *from* the centre, and for this case

$$V = \frac{f'(r-at)}{r} - \frac{f(r-at)}{r^2}.$$

At the same time equation (14) to the same degree of approximation becomes,

$$F(t) + P + \int \frac{dV}{dt}\, dr = 0.$$

And if $\rho = 1 + \delta$, $P = a^2 . \text{Nap.} \log(1 + \delta) = a^2 \delta$ nearly. Also

$$\frac{dV}{dt} = -a . \frac{f''(r - at)}{r} + a . \frac{f'(r - at)}{r^2}.$$

$$\text{Hence } \int \frac{dV}{dt} dr = -a \frac{f'(r - at)}{r},$$

$$\text{and } F(t) + a^2 \delta = a . \frac{f'(r - at)}{r}.$$

When the motion is directed to a fixed centre, these results apply to the whole of the fluid in motion, and at any time. When the centre is moving, r is in general a function of the co-ordinates of the point considered, and of the time, and the same results are applicable to all the points for which this function can be assigned. For instance, in the example considered above, in which the surface of displacement is that of a sphere of given radius, the centre of which moves in a straight line, r is at all times constant for all points of this surface. We have, therefore, from what is shewn above,

$$\frac{z - \gamma}{R} . \frac{d\gamma}{dt} = \frac{f'(R - at)}{R} - \frac{f(R - at)}{R^2},$$

an equation which is true whatever be t. Let $\frac{d\gamma}{dt}$, the velocity of the centre of the surface, be $m\phi(t)$, and let $\frac{z - \gamma}{R} = \cos\theta$, θ being the angle which the radius of a point whose co-ordinate is z makes with the line of motion. Also let $f(R - at) = f$. Then $f'(R - at) = -\frac{1}{a}\frac{df}{dt}$. Consequently for determining f we have the differential equation,

$$\frac{df}{dt} + \frac{fa}{R} + maR\cos\theta\phi(t) = 0.$$

By the integration of this equation f is obtained; whence $\frac{df}{dt}$ and consequently the density and the pressure at the surface of displacement are known. The question I have been considering is evidently identical with the Problem of the Resistance of the air to a vibrating sphere, of which I have given a solution in my last communication to this Society. But the method there employed for obtaining the above equation, requires the reasoning I have now gone through to render it complete.

18. Another inference may be drawn from equation (10) on the supposition that $Q = 0$ and V is very small. It thus becomes

$$a^2 \frac{d^2 V}{ds^2} - \frac{d^2 V}{dt^2} + a^2 \frac{dV}{ds}\left(\frac{1}{r} + \frac{1}{r'}\right) - Va^2\left(\frac{1}{r^2} + \frac{1}{r'^2}\right) = 0.$$

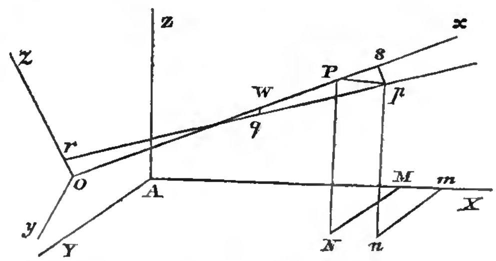
Now if r and r' be each infinitely great, the motion is strictly rectilinear, and must be the same at all points of any plane drawn perpendicular to the direction of propagation. But if r and r' be very large but not infinite, and if the motion be vibratory, we may conceive a portion of the fluid of the form of a cylinder to be alone agitated, whilst the rest of the fluid is stationary. The values of r and r' must, however, be infinitely great for points on the surface of the cylinder, and the velocity and condensation there must vanish. If the condensation be symmetrically disposed about the axis of the cylinder, the motion of particles situated on this axis will be rectilinear, but the vibrations of all other particles will be partly longitudinal and partly transversal. A line drawn at a given instant in the direction of the motion of the particles through which it passes will be of a serpentine form, approaching nearer to a straight line as r and r' are greater. That r and r' may be large, the diameter of the cylinder must be large compared to the breadth of an undulation. These considerations applied to the Undulatory Theory of Light, will account for the rectilinear propagation of a small cylindrical pencil of light without divergence.

CAMBRIDGE OBSERVATORY,
April 8, 1842.

ADDITIONAL NOTE.

THE following is a proof of the Proposition enunciated in Art. 6, viz. that when $udx + vdy + wdz$ is an exact differential the velocity does not vary from one point to another of a surface of displacement.

When the continuity of the fluid is maintained, the most general supposition that can be made respecting the directions of motion in each indefinitely small fluid element is, that they are normals to a surface of continuous curvature, and consequently that they pass ultimately through two focal lines perpendicular to their directions and situated in planes at right angles to each other.



Let, therefore, PWO, pqr be straight lines drawn in the directions of the motion at a given instant at two points P, p, of an indefinitely small element, and let them pass through the focal lines Wq, Or. The point P is referred to the rectangular axes AX, AY, AZ. $AM = X$, $MN = Y$, $NP = Z$. Pp is drawn parallel to AX and is equal to δX. Let $OW = l$, $WP = r$, and draw ps perpendicularly on OWP. Then $Ps = \delta r$. Take now another system of rectangular axes Ox, Oy, Oz, of which Ox coincides with OWP, Oz with the focal line Or, and Oy is parallel to Wq. Let $Or = h$, $Wq = k$. Suppose the equations of Pp referred to the axes Ox, Oy, Oz, to be $x = az + \alpha$, $y = bz + \beta$. Because it passes through P, the co-ordinates of which are $x = l + r$, $y = 0$, $z = 0$, it follows that $\alpha = l + r$, and $\beta = 0$. Hence the equations become $x = az + l + r$ and $y = bz$.

Again, let the equations of pqr be $x = mz + n$, $y = pz + q$. As this line passes through the point r, the co-ordinates of which are $x = 0$, $y = 0$, $z = h$, we have $0 = mh + n$ and $0 = ph + q$. Therefore $x = m(z - h)$, and $y = p(z - h)$. Since also the line passes through q, the co-ordinates of which are $x = l$, $y = k$, $z = 0$, we have $l = n$. Hence $mh = -l$, and $h = -\frac{l}{m}$. The equations of pqr thus become $x = mz + l$, $y = p\left(z + \frac{l}{m}\right)$. Now the co-ordinate Os of the point p is $l + r + \delta r$. Hence, the first equation of Pp gives $l + r + \delta r = az + l + r$, or $z = \frac{\delta r}{a}$, and consequently from the second, $y = \frac{b}{a} . \delta r$. These are the other two co-ordinates of the point p. By substituting these values of the co-ordinates in the equations of the line pqr, which passes through p, we shall readily find that $m = a + \frac{ar}{\delta r}$, and $p = \frac{b(r + \delta r)}{l + r + \delta r}$.

If now $x = a'z$, $y = b'z$ be the equations of a line drawn through O parallel to AZ, the cosine of the $\angle NPO = \frac{a'}{\sqrt{1 + a'^2 + b'^2}}$; and the cosine of the $\angle npr = \frac{1 + ma' + pb'}{\sqrt{1 + a'^2 + b'^2}\sqrt{1 + m^2 + p^2}}$. Hence by substituting the values of m and p, expanding to the first power of δr, and remembering that $1 + aa' + bb' = 0$, it will be found that,

$$\cos \angle apr = \frac{a'}{\sqrt{1 + a'^2 + b'^2}} . \left(1 + \frac{l + r + bb'r}{aa'r(l + r)} \delta r\right).$$

Let V and $V + dV$ be the velocities at P and p at the same instant, and let w and $w + \delta w$ be their components in the direction of the axis of z. Then

$$w = V \cos \angle NPO = \frac{Va'}{\sqrt{1 + a'^2 + b'^2}},$$

$$\text{and } w + \delta w = (V + \delta V) \cos \angle npr = \frac{(V + \delta V)a'}{\sqrt{1 + a'^2 + b'^2}} . \left(1 + \frac{l + r + bb'r}{aa'r(l + r)} . \delta r\right).$$

$$\text{Consequently, } \delta w = \frac{V(l + r + bb'r)\delta r}{ar(l + r)\sqrt{1 + a'^2 + b'^2}} + \frac{a'\delta V}{\sqrt{1 + a'^2 + b'^2}} \text{ ultimately.}$$

Also, $\cos \angle pPs = \frac{a}{\sqrt{1 + a^2 + b^2}}$, and $\delta X = \frac{\delta r}{a}\sqrt{1 + a^2 + b^2}$. Hence

$$\frac{\delta w}{\delta X} = \frac{\frac{V}{r} + \frac{Vbb'}{l+r} + \frac{aa'\delta V}{\delta r}}{\sqrt{(1 + a^2 + b^2)(1 + a'^2 + b'^2)}}.$$

If therefore the variation of velocity δV from P to p be the same as from P to s, that is, if the variation be nothing from p to s, the limit of the ratio $\frac{\delta V}{\delta r}$, is the differential coefficient $\frac{dV}{dr}$, taken as if the variation were from one point to another of the line OWP. Hence,

$$\frac{dw}{dX} = \frac{\frac{V}{r} + \frac{Vbb'}{l+r} + aa'.\frac{dV}{dr}}{\sqrt{(1 + a^2 + b^2)(1 + a'^2 + b'^2)}}.$$

The value of $\frac{du}{dZ}$ is evidently derived from that of $\frac{dw}{dX}$ by interchanging a, a', and b, b'; and since, when this is done, the right-hand side of the equation is unaltered, we conclude that $\frac{dw}{dX} = \frac{du}{dZ}$.

So $\frac{dw}{dY} = \frac{dv}{dZ}$, and $\frac{du}{dY} = \frac{dv}{dX}$.

Consequently, $udX + vdY + wdZ$ is an exact differential.

Cambridge Observatory,
May 27, 1842.

XXII. *On the Propagation of Luminous Waves in the interior of Transparent Bodies.* *By the Rev.* M. O'BRIEN, M.A., *late Fellow of Caius College, Cambridge.*

[Read *April* 25, 1842.]

§ 1. THE chief object of the following Paper is to investigate what effects may be due to the action of the material particles upon the etherial, in the case of light transmitted through a transparent body; and among other things to shew that the dispersion of light in passing through a prism may be acounted for without having recourse to the Hypothesis of Finite intervals*.

The following is a brief statement of the results arrived at, some of which, if true, must be of considerable importance in the Undulatory Theory of Light.

§ 2. I have endeavoured to prove:—

(1°) That the action of the material particles upon the etherial is very nearly the same as if the former particles were absolutely fixed. (See § 14.)

(2°) That this action introduces certain simple terms, viz.:—

$$\frac{m_{\prime}C}{m}\alpha, \quad \frac{m_{\prime}C}{m}\beta, \quad \frac{m_{\prime}C}{m}\gamma,$$

into the equations of motion of the etherial particles. (See § 15.)

* By the Hypothesis of Finite intervals, I mean the hypothesis that the particles of ether are placed at intervals which are finite compared with λ, the length of a luminous wave; or, what amounts to the same thing, that the intensity of the molecular force exercised by one particle upon another is not very small at the distance λ.

I may here observe, however, that the present paper does not in any way assume that this hypothesis is not true, as will appear in Art. 12; it only attempts to shew that the undulatory theory can do without it. In one place, indeed, I have endeavoured to shew that the Hypothesis of Finite intervals is very improbable.

(3°) That the law of molecular force is not likely to be such as to make C zero in all cases. (See § 16.)

(4°) I have investigated the condition necessary in order that the equilibrium of the etherial particles may be stable. (See § 16.)

(5°) Also the conditions necessary in order that the vibrations of the ether may be wholly transversal or wholly normal. (See § 18, 19.)

(6°) I have shewn that the equations of motion may be each solved separately when the vibrations are wholly transversal or wholly normal. (See § 20, 21.)

(7°) That transversal and normal vibrations are in general propagated with different velocities. (See § 25.)

(8°) That the velocity of transversal vibrations may be of any magnitude, the propagation of normal vibrations impossible, and yet the equilibrium stable; and here I have shewn that the law of molecular force cannot be the inverse fourth power, nor the inverse square, if the theory of transverse vibrations be true. (See § 26.)

(9°) That the action of the material upon the etherial particles produces an alteration in the velocity of light which is different for different colours. (See § 28.)

(10°) I have investigated the additional alteration which the *motion* of the particles of matter produces in the velocity of light, and shewn that it must be extremely small, and that it does not depend on the length of the wave if C be zero; and that consequently the *motion* of the material particles cannot produce dispersion independently of their *direct action* on the particles of ether. (See § 29.)

(11°) I have shewn that light cannot be propagated with a uniform velocity in transparent bodies, unless the particles vibrate according to the cycloidal law. (See § 33.)

(12°) That the consequence of this must be a dispersion of *homogeneous* light in passing through a prism, when the vibrations are not cycloidal. (See § 35.)

(13°) I have calculated the amount of this dispersion, and shewn how we may determine experimentally whether the vibrations of light are cycloidal or not. (See § 38.)

(14°) I have shewn that the variation of the velocity of propagation may lead to the formation of dark lines in the spectrum, but I have not attempted to pursue this subject into detail in the present paper.

(15°) In the course of the paper I have introduced the hypothesis of symmetrical arrangement, merely for the purpose of avoiding complexity, but at the conclusion of the paper I have abandoned this hypothesis, and shewn, I think, by simple reasoning, that the results previously obtained on the supposition of symmetrical arrangement are true when the arrangement is not symmetrical, in consequence of the action of the material particles on the etherial. This part of the subject is of considerable importance, as I hope to shew fully hereafter, and most probably leads to an explanation of the absorption of light by transparent bodies, and the natural colours of bodies. (See § 40, &c.)

All these results, so far as I am aware, are new, except the seventh, which the late *Mr Green* announced in the Cambridge Philosophical Transactions, Vol. VII. Part I., without, however, proving it. Since I read the present paper before the Society, I have been informed that *M. Cauchy* has arrived at the same result.

§ 3. I must here notice a Memoir by *M. G. Lamé*, in the *Journal de l'Ecole Polytechnique, Tome XIV*, in which his object is to investigate the effect produced by the action of the material particles on the undulations of the ether *considered as a continuous fluid mass*. The analysis he makes use of is extremely complicated, and is altogether different from that employed in the present paper. There is one of his results which appears to be the same as one of mine, but really is not: it is this, that the force exercised by the particles of matter on the ether produces an alteration in the velocity of light, which gives rise to dispersion: he obtains this result on the express condition, that the force exerted by the particles of matter varies as $(\text{distance})^{-2}$. Now I have shewn (if my reasoning be correct) that there is no alteration produced in the velocity of light when the force exerted by the particles of matter varies as $(\text{distance})^{-2}$: and my explanation of dispersion falls to the ground if the law of molecular force be $(\text{the distance})^{-2}$.

I think therefore, that I am fully justified in saying, that M. Lamé has not anticipated me in the explanation I have given of dispersion; indeed, a cursory reading of the two papers is quite sufficient to shew that there is no resemblance between them in principle or detail*.

ANALYTICAL INVESTIGATION.

§ 4. I suppose that the etherial medium consists of a set of discrete particles, that one particle exerts upon another a force which acts in the line joining the particles, and is some function of the distance between them; and I suppose, that in transparent bodies the particles of matter and the etherial particles act upon each other in a similar manner.

§ 5. *To form the equations of motion of the etherial medium as it exists in transparent bodies.*

Let $\left.\begin{matrix} x, y, z, \\ x', y', z', \end{matrix}\right\}$ be the co-ordinates of any two particles of ether, P and P', $\left.\begin{matrix} x_{,}, y_{,}, z_{,}, \end{matrix}\right\}$ the co-ordinates of any particle $P_{,}$ of the transparent body, when in a state of equilibrium.

$\left.\begin{matrix} x + \alpha, & y + \beta, & z + \gamma, \\ x' + \alpha', & y' + \beta', & z' + \gamma', \\ x_{,} + \alpha_{,}, & y_{,} + \beta_{,}, & z_{,} + \gamma_{,}, \end{matrix}\right\}$ the co-ordinates of the same particles respectively at any time t during the motion.

r the distance between P and P', when in a state of equilibrium.
r' ... do. .... do. ... P and $P_{,}$, when in a state of equilibrium.

$\left.\begin{matrix} r + \rho, \\ r' + \rho', \end{matrix}\right\}$ the distances between the same particles respectively at the time t.

* M. Lamé has also attempted to account for the formation of the dark lines in the spectrum, but he assigns a cause very different from that which I have shewn may produce these remarkable interruptions.

$m\ r f(r)$ the force of P' on P,

$m_, r' \phi(r')$ ... do. ... of $P_,$ on P.

Then the force of P' on P at the time t resolved parallel to the axis of x is

$$m(r+\rho)f(r+\rho)\frac{x'+a'-x-a}{r+\rho}, \text{ which } = mf(r+\rho)(x'+a'-x-a),$$

and we have similar expressions for the resolved parts of the force of $P_,$ on P; hence we evidently have

$$\left.\begin{array}{l}\dfrac{d^2 a}{dt^2} = \Sigma m f(r+\rho)(x'+a'-x-a) + \Sigma_, m_, \phi(r'+\rho')(x_, + a_, - x - a) \\ \qquad\qquad \text{and similar expressions for } \dfrac{d^2\beta}{dt^2}\ \dfrac{d^2\gamma}{dt^2}\end{array}\right\} \ldots (A.)$$

The sign of summation Σ refers to all the particles of ether, and $\Sigma_,$ to all the particles of matter, which exert sensible forces on P.

§ 6. *To simplify these equations when the motion is a very small vibratory motion.*

For the sake of brevity assume δx, δa, to denote $x'-x$, $a'-a$, respectively, and a similar notation with respect to the other co-ordinates; also assume Δx, Δa, to denote $x_, - x$, $a_, - a$, respectively, and a similar notation with respect to the other co-ordinates.

Since the motion is a very small vibratory motion, we may assume that the relative displacement of any two particles is very small, compared with the distance between them.

This assumption, so far as I am aware, has been made by every one who has written on the subject of undulations, whether in the case of light or of sound; indeed the equation of continuity in Hydrodynamics cannot be proved unless this assumption is made. But it seems to me to be no assumption in the present investigation, at least if we confine our attention to the case of light at some distance from its source; for example, solar light at the Earth: for suppose that there is a considerable degree of condensation and rarefaction in the ether in the immediate vicinity of the Sun, be it ever so considerable there, it is clear that it must be very small at the Earth; for it must vary inversely as the square of the distance from the Sun; hence we are safe in supposing that there is very little condensation and rarefaction in the undulations of ether which constitute the solar light we have to do with; and if the vibrations be transversal, as we have every reason to suppose them to be, this is still more evident; for transversal vibrations cannot be propagated unless the variation of the density of the ether caused by the motion be extremely small.

If there be very little condensation or rarefaction in the ether, it is clear that the relative motion of any two contiguous particles must be very small, compared with their actual distance from each other. Indeed if this be not true, the principle of the superposition of small motions cannot be applied to the etherial undulations, and the whole undulatory theory must fall to the ground; moreover, the velocity of light in vacuum cannot be uniform, as we know it to be. Hence I think that there is just the same degree of assumption in supposing that the relative motions of the etherial particles are very small compared with their actual distances, as there is in supposing that light consists in a succession of undulations. I make these remarks because I have heard objections urged against the simplification I am now about to make in the equations of motion, and which has been made by every author I am acquainted with, under similar circumstances.

Proceeding then upon the assumption, if it may be so called, that the relative motion of two contiguous particles must be very small, compared with their actual distance from each other, it is evident that $\delta\alpha$, $\delta\beta$, $\delta\gamma$, and ρ, must be very small, compared with δx, δy, δz, and r; hence since

$$r^2 = \delta x^2 + \delta y^2 + \delta z^2, \text{ and } (r+\rho)^2 = (\delta x + \delta\alpha)^2 + (\delta y + \delta\beta)^2 + (\delta z + \delta\gamma)^2,$$

we have very nearly, (subtracting, and dividing by $2r$,)

$$\rho = \frac{1}{r}(\delta x\,\delta\alpha + \delta y\,\delta\beta + \delta z\,\delta\gamma);$$

therefore since $f(r+\rho)(\delta x + \delta\alpha) = \{f(r) + \rho f'(r)\}(\delta x + \delta\alpha)$, we have (neglecting $\rho\delta\alpha$ and putting for ρ its value)

$$f(r+\rho)(\delta x+\delta\alpha)=f(r)(\delta x+\delta\alpha)+\frac{1}{r}f'(r)\,\delta x\,(\delta x\,\delta\alpha+\delta y\,\delta\beta+\delta z\,\delta\gamma).$$

Hence, transforming in the same manner the similar quantities in the equation (A), and observing that by the condition of previous equilibrium we have

$$\Sigma m f(r)\,\delta x+\Sigma_{,}m_{,}\phi(r')\,\Delta x=0,$$

the equation (A) becomes

$$\left.\begin{aligned}\frac{d^2\alpha}{dt^2}&=\Sigma m\left\{f(r)\,\delta\alpha+\frac{1}{r}f'(r)\,\delta x\,(\delta x\,\delta\alpha+\delta y\,\delta\beta+\delta z\,\delta\gamma)\right\}\\&+\Sigma_{,}m_{,}\left\{\phi(r')\,\Delta\alpha+\frac{1}{r'}\phi'(r')\,\Delta x\,(\Delta x\,\Delta\alpha+\Delta y\,\Delta\beta+\Delta z\,\Delta\gamma)\right\}\end{aligned}\right\}\ldots\ldots(B).$$

§ 7. I shall now proceed to put these equations in the form of partial differential equations, in order to make them more manageable. I must first observe, that $\alpha, \beta, \gamma, \alpha_{,}, \beta_{,}, \gamma_{,}, x, y, z, x_{,}, y_{,}, z_{,}$, &c. are quantities which, at first sight, do not appear capable of continuous variation, since they belong to a set of discontinuous points: but notwithstanding this, we are evidently quite at liberty to look upon these quantities as continuous variables; for instance, we may suppose that α is a continuous function of x, y, z, t, having the proper value when x, y, z become the co-ordinates of any particle of ether: for, originally, we only assumed that α has certain values when x, y, z belong to any particle of ether, but we made no assumption whatever respecting the values of α when x, y, z do not belong to any particle of ether, and therefore we may suppose these values such, that α shall be a continuous function of x, y, z, t for all values of x, y, z, and the same is true of β, γ, &c. It is just on the same principle that we may draw a continuous curve through any series of detached points.

In reducing the equation (B) to the form of a partial differential equation, I shall first omit the part under the sign $\Sigma_{,}$, and afterwards restore it; this will be found the simplest course to pursue.

§ 8. *To put the equation* (B) *in the form of a partial differential equation, omitting the part under the sign* $\Sigma_{,}$.

As we have just explained, α' may be considered to be a continuous function of x', y', z', t; and α is its value when $x' = x$, $y' = y$, $z' = z$; hence, by Taylor's Theorem, we may suppose $\alpha' - \alpha$ or $\delta\alpha$ expanded in the form

$$\frac{d\alpha}{dx}\delta x + \frac{d\alpha}{dy}\delta y + \frac{d\alpha}{dz}\delta z + \frac{d^2\alpha}{dx^2}\frac{\delta x^2}{2} + \frac{d^2\alpha}{dx\,dy}\delta x\,\delta y + \frac{d^2\alpha}{dy^2}\frac{\delta y^2}{2} + \text{\&c. \&c.};$$

if we substitute this value of $\delta\alpha$ in equation B, $\frac{d\alpha}{dx}$ $\frac{d\alpha}{dy}$ $\frac{d\alpha}{dz}$ $\frac{d^2\alpha}{dx^2}$ &c. may all be brought outside Σ, and in the same manner, if we substitute similar expressions for $\delta\beta$ and $\delta\gamma$, the partial differential coefficients of β and γ may be brought outside Σ: the result of these substitutions will evidently be a linear equation between $\frac{d^2\alpha}{dt^2}$, and the successive partial differential coefficients of α, β, γ with respect to x, y, z, multiplied by such quantities as

$$\Sigma m f(r)\,\delta x, \qquad \Sigma m f(r)\,\delta x^2, \qquad \Sigma m f(r)\,\delta x\,\delta y, \text{ \&c. \&c.}:$$

these quantities will evidently be, in general, different for different particles; that is, they will be functions of x, y, z; hence the equation (B) will, in general, become a linear differential equation with variable coefficients.

We cannot determine what functions of x, y, z these coefficients are, since to do so we ought to know the law of force of one particle on another, and the manner in which the particles are arranged when in a state of equilibrium, neither of which things we know; hence it appears impossible to make use of the equation (B) unless we employ some hypothesis to simplify it.

§ 9. The hypothesis which naturally suggests itself is, that of a symmetrical arrangement of the particles when in a state of equilibrium. But there is a difficulty here, arising from the influence that must be exerted by the material particles on the arrangement of the etherial particles; for supposing the material particles symmetrically arranged, it is evident that if there be a number of etherial particles surrounding each material particle, the arrangement of the former cannot be symmetrical; for they will be disturbed from their positions of symmetry by the forces exercised on them by the material particles. Hence, if there be a number of etherial particles surrounding each material particle, it is impossible, in general, that both sets of particles can be symmetrically arranged.

But let us suppose that there are not so many, or at most, as many etherial particles as there are material, then it is evident that if the material particles be symmetrically arranged, so also will the etherial. The following figures will make this evident. Figure (4) represents what we may conceive to be the arrangement of the particles when each material particle is surrounded by several etherial; the large dots representing the former, and the small dots the latter. Figure (3) represents what may be the arrangement when there are fewer particles of ether than of matter; and Figure (2) when there are just the same number of both. Figure (1) represents what, I think, is not at all an improbable arrangement in the case of very transparent bodies, where, though there are more particles of ether than of matter, yet the arrangement is symmetrical, in consequence of the etherial particles being repelled so strongly by the material that they form themselves into globules, which may be regarded each as one particle: I shall hereafter explain on what grounds I conceive this to be a very probable arrangement in the case of very transparent bodies.

Fig. (1.)

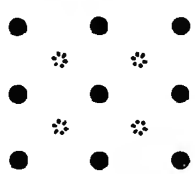


Fig. (2.)

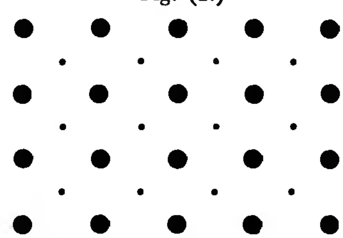


Fig. (3.)

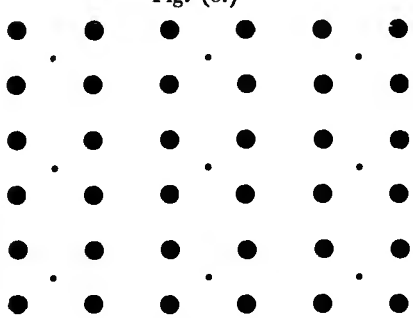
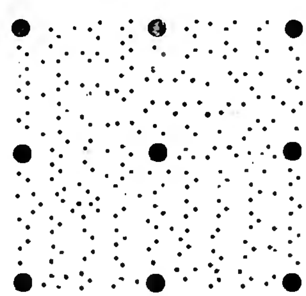


Fig. (4.)



§ 10. I shall now make use of the hypothesis of symmetrical arrangement to simplify the equation (B), which hypothesis, as I have shewn, necessarily implies that there are not so many, or at most, as many particles of ether as there are of matter, or that the etherial particles are formed into globules by the repulsion of the material particles. ***But I make use** of this hypothesis merely for the present* in order to make myself more readily understood. I shall hereafter prove that the results obtained by means of this hypothesis are equally true when the arrangement of the etherial particles is unsymmetrical, in consequence of the influence exerted on them by the material particles.

§ 11. *To simplify the partial differential equations by means of the hypothesis of the symmetrical arrangement of the etherial and material particles.*

If the particles be all arranged symmetrically, it is evident that we may so assume the axes of co-ordinates that the arrangement of the particles shall be symmetrical with respect to them. This being the case, it is evident that, if $F(r)$ be any function of r, $\Sigma m\{F(r)\delta x^p \delta y^q \delta z^s\}$ is zero *unless* each of the indices p q and s be even, and if each of these be *even*, then this sum is the same for every etherial particle, i. e. it is a constant. Moreover, p q s may evidently be interchanged without altering the value of this sum.

Hence if we put

$$mM = \Sigma m\{f(r)\delta x^2\}, \quad mN = \Sigma m\left\{\frac{1}{r}f'(r)\delta x^2\delta y^2\right\}, \quad mP = \Sigma m\left\{\frac{1}{r}f'(r)\delta x^4\right\},$$

the partial differential equation becomes (omitting at present the part under the sign $\Sigma_{,}$)

$$\frac{1}{m}\frac{d^2\alpha}{dt^2} = \frac{M}{2}\left(\frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2}\right) + \frac{P}{2}\frac{d^2\alpha}{dx^2},$$

$$+ \frac{N}{2}\left(\frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2} + 2\frac{d^2\beta}{dx\,dy} + 2\frac{d^2\gamma}{dx\,dz}\right),$$

+ differential coefficients of the 4th and higher orders.

Now there is a very simple relation between P and N: for put for a moment $\delta x = u\cos\theta$, $\delta y = u\sin\theta$, then (in virtue of the symmetry of the arrangement) for each value of u and r, θ admits of a set of equidifferent values whose sum is 2π; therefore

$$\Sigma\frac{1}{r}f'(r)\,\delta x^4 = \Sigma\frac{1}{r}f'(r)u^4\cos^4\theta = \Sigma\frac{1}{r}f'(r)\frac{u^4}{8}(3 + 4\cos 2\theta + \cos 4\theta),$$

$$= \frac{3}{8}\Sigma\frac{1}{r}f'(r)\,u^4,$$

and $$\Sigma\frac{1}{r}f'(r)\,\delta x^2\delta y^2 = \Sigma\frac{1}{r}f'(r)u^4\cos^2\theta\sin^2\theta = \Sigma\frac{1}{r}f'(r)\frac{u^4}{8}(1 - \cos 4\theta),$$

$$= \frac{1}{8}\Sigma\frac{1}{r}f'(r)\,u^4;$$

hence $P = 3N$.

Hence if for brevity we put $\frac{M+P}{2} = A$, $\frac{M+N}{2} = B$, and therefore $A - B = \frac{P}{2} - \frac{N}{2} = N$, the differential equation becomes

$$\left.\begin{array}{l}\dfrac{d^2\alpha}{dt^2} = mA\dfrac{d^2\alpha}{dx^2} + mB\left(\dfrac{d^2\alpha}{dy^2} + \dfrac{d^2\alpha}{dz^2}\right) + m(A - B)\dfrac{d}{dx}\left(\dfrac{d\beta}{dy} + \dfrac{d\alpha}{dx}\right)\\ \quad + \text{ differential co-efficients of the } 4^{\text{th}} \text{ and higher orders.}\end{array}\right\}.$$

Where,

$$A = \frac{1}{2}\Sigma\left\{f(r)\,\delta x^2 + \frac{3}{r}f'(r)\,\delta x^2\delta y^2\right\},$$

$$B = \frac{1}{2}\Sigma\left\{f(r)\,\delta x^2 + \frac{1}{r}f'(r)\,\delta x^2\delta y^2\right\}.$$

It is easy to see from what has been proved, that in these expressions we may put $\frac{r^2}{3}$ for δx^2, and $\frac{r^4}{15}$ for $\delta x^2\delta y^2$, and the values of A and B will be unaltered. Hence, if $R = rf(r)$ it is evident that we have the following simple expressions for A and B,

$$A = \frac{1}{10}\Sigma\left\{r^{\frac{4}{3}}\frac{d(Rr^{\frac{2}{3}})}{dr}\right\},\quad B = \frac{1}{30}\Sigma\left\{\frac{1}{r^2}\frac{d(Rr^4)}{dr}\right\};$$

R here represents the law of molecular force.

§ 12. *To compare the magnitudes of the several terms which compose this equation; supposing the vibrations of the ether to constitute a common wave of light whose length is* λ.

It is evident that on this supposition α, β, γ will be of some such form as $c \sin \frac{2\pi}{\lambda}(vt - px - qy - sz)$ (where $p^2 + q^2 + s^2 = 1$); hence the second differential coefficients of α, β, γ will evidently be of the same order of magnitude as $\frac{c}{\lambda^2}$, the fourth differential coefficients as $\frac{c}{\lambda^4}$, and so on.

Again, it is evident that δx, δy, δz are of the same order of magnitude as r, and $rf'(r)$ as $f(r)$ (for if we suppose $f(r) = Mr^m + Nr^n + \&c.$ then $rf'(r) = mMr^m + nNr^n + \&c.$), hence in the equation (6) the part involving second differential coefficients is of the same order of magnitude as $\Sigma f(r) \frac{r^2}{\lambda^2} c$; the part involving fourth differential coefficients as $\Sigma f(r) \frac{r^4}{\lambda^4} c$, and so on.

Now we know that the molecular forces of all ordinary bodies are quite insensible at the smallest distances that can be measured; and therefore they must be so at the distance λ, which, though small, is yet measurable; hence we may suppose that the particles of ether exercise no sensible force at the distance λ, and this being the case, $\Sigma f(r) \frac{r^4}{\lambda^4} c$ must be extremely small compared with $\Sigma f(r) \frac{r^2}{\lambda^2} c$, since r is the distance between two particles which exercise a sensible force on each other. I think we are quite justified in this supposition by analogy, especially if we consider how much more minute the ethereal particles must be than those of matter; for I can scarcely conceive that the delicate particles of ether can exercise a sensible force at a greater distance than the comparatively gross particles of matter do. Besides, observation seems to shew that all colours are propagated with equal velocity in vacuum: now if the particles of ether exercise a sensible force at the distance λ this cannot be the case, for then that part of the equation which involves fourth and higher differential coefficients cannot be neglected, and it is well known that if that part of the equation be sensible, different colours must be propagated with different velocities; hence if it be true, as it most probably is, that all colours are propagated with the same velocity in vacuum, the particles of ether cannot exercise any sensible force at the distance λ.

If this be correct, the explanation of the phenomenon of dispersion by the hypothesis of finite intervals falls to the ground; for if the particles of ether in the interior of transparent bodies are placed at intervals not extremely small compared with λ, they must exercise very little force upon each other, and therefore the ethereal medium in the interior of transparent bodies must be almost devoid of elasticity, which evidently cannot be the case. But even supposing the hypothesis of finite intervals to be true, I may in the present investigation neglect the terms of the equation involving fourth and higher differential coefficients; for experiment shews that the dispersion of a ray is small compared with the whole deviation produced by refraction; therefore these terms (if not quite insensible) must be small compared with those involving second differential coefficients. Now it is not my object to investigate that part of the dispersion (if there be any) which arises from these terms, but that which arises from other terms; namely, those introduced into the equations in consequence of the influence exerted by the particles of matter on those of ether; therefore, in accordance with a well-known principle, I may neglect the former terms in investigating the effect of the latter.

§ 13. Neglecting, then, the terms in the equation which involve fourth and higher differential coefficients for the reasons just stated, we have the following equations of motion for any particle of ether, (writing down the two other equations for the motion parallel to the axes of y and z;)

$$\left.\begin{aligned}
\frac{1}{m}\frac{d^2\alpha}{dt^2} &= A\frac{d^2\alpha}{dx^2} + B\left(\frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2}\right) + (A-B)\frac{d}{dx}\left(\frac{d\beta}{dy} + \frac{d\gamma}{dz}\right)\\
\frac{1}{m}\frac{d^2\beta}{dt^2} &= A\frac{d^2\beta}{dy^2} + B\left(\frac{d^2\beta}{dx^2} + \frac{d^2\beta}{dz^2}\right) + (A-B)\frac{d}{dy}\left(\frac{d\alpha}{dx} + \frac{d\gamma}{dz}\right)\\
\frac{1}{m}\frac{d^2\gamma}{dt^2} &= A\frac{d^2\gamma}{dz^2} + B\left(\frac{d^2\gamma}{dx^2} + \frac{d^2\gamma}{dy^2}\right) + (A-B)\frac{d}{dz}\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy}\right)
\end{aligned}\right\} (C).$$

These are the equations * of motion of the etherial particles, omitting the terms depending on the action of the material particles, consequently these are the equations of motion of the ether as it exists in vacuum.

§ 14. *To compare the action of the material upon the etherial particles with what it would be if the former were absolutely fixed.*

In the equation (*B*) the part which arises from the action of the material particles, i. e. the part under the sign $\Sigma_{,}$ may be written thus, putting for $\Delta\alpha$, $\Delta\beta$, $\Delta\alpha$, their values $\alpha_{,} - \alpha$, $\beta_{,} - \beta$, $\gamma_{,} - \gamma$), viz.

$$\Sigma_{,} m_{,} \left\{ \phi(r')(\alpha_{,} - \alpha) + \frac{1}{r'}\phi'(r')\Delta x \{\Delta x(\alpha_{,} - \alpha) + \Delta y(\beta_{,} - \beta) + \Delta z(\gamma_{,} - \gamma)\} \right\}.$$

Now let α_2 β_2 γ_2 be the greatest values which $\alpha_{,}$ $\beta_{,}$ $\gamma_{,}$ respectively admit of at the time t, and α_3 β_3 γ_3 the least; then observing, that α β γ, α_2 β_2 γ_2, α_3 β_3 γ_3, may be brought outside the sign $\Sigma_{,}$, and that $\Sigma_{,} m_{,} \frac{1}{r'}\phi'(r')\Delta x\Delta y$, and $\Sigma_{,} m_{,} \frac{1}{r'}\phi'(r')\Delta x\Delta z$ are zero in consequence of the symmetry, it is manifest that the above expression lies between

$$(\alpha_2 - \alpha)\Sigma m_{,}\left(\phi(r') + \frac{1}{r'}\phi'(r')\Delta x^2\right) \text{ and } (\alpha_3 - \alpha)\Sigma m_{,}\left(\phi(r') + \frac{1}{r'}\phi'(r')\Delta x^2\right),$$

or between $m_{,}C(\alpha_2 - \alpha)$ and $m_{,}C(\alpha_3 - \alpha)$,

if we assume C to denote $\Sigma_{,}\left\{\phi(r') + \frac{1}{r'}\phi'(r')\Delta x^2\right\}$.

Now in the case of common luminous waves passing through transparent bodies, the particles of matter are put in motion solely by the vibrations of the particles of ether: if we consider how extremely

* These equations are obtained by M. Cauchy in his *Exercices*, Vol. III. by a complicated method, different from that made use of in the present paper. Mr. Green, also, has obtained the same equations, in the Cambridge Philosophical Transactions, Vol. VII. Part I. by a very general but complicated method.

large the masses of the former must be compared with those of the latter*, it is evident that the motion of the former must be small compared with that of the latter: hence, α_2 and α_3 must in general be small compared with α; and therefore $m_{,}C(\alpha_2 - \alpha)$ and $m_{,}C(\alpha_3 - \alpha)$ must be nearly the same as $-m_{,}C\alpha$, and therefore, *à fortiori*, the expression under the sign $\Sigma_{,}$, which we know always lies between $m_{,}C(\alpha_2 - \alpha)$ and $m_{,}C(\alpha_3 - \alpha)$, must in general be nearly the same as $-m_{,}C\alpha$, and this is evidently true, however small or large C may be. Hence we are justified in proceeding upon the following supposition; viz., *that the action of the particles of matter on that of ether in the case of common luminous waves passing through transparent bodies, is very little altered by the motion of the material particles; and that therefore in estimating that action we may suppose the particles of matter absolutely fixed for a first approximation.*

§ 15. *Hence it appears, that the equations of motion of the etherial particles, taking into account the action of the material particles upon them, are as follows, at least for a first approximation.*

$$\left.\begin{aligned}\frac{1}{m}\frac{d^2\alpha}{dt^2} &= A\frac{d^2\alpha}{dx^2} + B\left(\frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2}\right) + (A - B)\frac{d}{dx}\left(\frac{d\beta}{dy} + \frac{d\gamma}{dx}\right) - \frac{m_{,}}{m}C\alpha \\ \frac{1}{m}\frac{d^2\beta}{dt^2} &= A\frac{d^2\beta}{dy^2} + B\left(\frac{d^2\beta}{dx^2} + \frac{d^2\beta}{dz^2}\right) + (A - B)\frac{d}{dy}\left(\frac{d\alpha}{dx} + \frac{d\gamma}{dy}\right) - \frac{m_{,}}{m}C\beta \\ \frac{1}{m}\frac{d^2\gamma}{dt^2} &= A\frac{d^2\gamma}{dz^2} + B\left(\frac{d^2\gamma}{dx^2} + \frac{d^2\gamma}{dy^2}\right) + (A - B)\frac{d}{dz}\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy}\right) - \frac{m_{,}}{m}C\gamma\end{aligned}\right\} (D).$$

$$\text{where } C = \Sigma\left\{\phi(r') + \frac{1}{r'}\phi'(r')\,\Delta x^2\right\}.$$

The terms $\frac{m_{,}}{m}C\alpha$, $\frac{m_{,}}{m}C\beta$, $\frac{m_{,}}{m}C\gamma$, which I have thus introduced into the equations, lead to very remarkable results, as will appear presently.

§ 16. *To shew that the law of molecular force is not likely to be such as to make* C *zero in all cases.*

* That this is the case, I think the very small resistance experienced by the heavenly bodies from the ethereal medium seems to prove.

In any system of particles arranged symmetrically, suppose that one of the particles is slightly displaced from its position of rest, the others remaining undisturbed; then if we make use of the notation in Article (1), putting $\alpha' = 0$, $\beta' = 0$, $\gamma' = 0$, and therefore $\delta\alpha = -\alpha$, $\delta\beta = -\beta$, $\delta\gamma = -\gamma$; the forces which act on the particle parallel to the axes in consequence of its displacement, are evidently $-C\alpha$, $-C\beta$, $-C\gamma$,

$$\text{where } C = \Sigma m \{f(r) + \frac{1}{r} f'(r)\, \delta x^2\}.$$

Hence if the law of molecular force be such that C is zero in all cases, it is evident that in a symmetrical system any particle may be slightly disturbed from its position of rest, without bringing any force into action upon it, i. e. the system is in neutral equilibrium*.

Now it is very improbable that substances in nature are so held together, that a particle may be slightly displaced from its position of rest without bringing any force into action upon it: therefore it is not likely that the molecular force is of such a nature as to make C zero in all cases.

If the system consist of two or more sets of different particles exercising different kinds of molecular forces, the same is evidently true; for then the forces which act on the particle parallel to the axes, in consequence of its displacement, are $-\alpha\Sigma C$ $-\beta\Sigma C$ $-\gamma\Sigma C$,

where

$$\Sigma C = \{\Sigma m\, [f(r) + \frac{1}{r} f'(r)\, \delta x^2] + \Sigma_{\prime} m_{\prime} [\phi(r_{\prime}) + \frac{1}{r'} \phi'(r_{\prime})\, \Delta x^2] + \text{similar terms}\},$$

* Hence it is evident *that the condition of stability of the equilibrium of the ether in vacuum is*

$$\Sigma \{f(r) + \frac{1}{r} f'(r)\, \delta x^2\} = \text{a positive quantity},$$

and in the interior of a transparent body

$$m \Sigma \{f(r) + \frac{1}{r} f'(r)\, \delta x^2\} + m_{\prime} \Sigma_{\prime} \{\phi(r') + \frac{1}{r'} \phi'(r')\, \Delta x^2\} = \text{a positive quantity}.$$

As in § 11, the condition of stability in vacuum may be put in the form

$$\Sigma \left\{\frac{1}{r^3} \frac{d(Rr^3)}{dr}\right\} = \text{a positive quantity},$$

R being the law of molecular force.

which forces are evidently zero, if the law of molecular force be such as make C zero in all cases.

If the law of molecular force be the inverse square of the distance it is easy to see that C must be zero, no matter whether the force be attractive or repulsive.

For suppose that the molecular force (i.e $mrf(r)$) equals $\pm\frac{m}{r^2}$, then

$$\Sigma m\left\{f(r)+\frac{1}{r}f'(r)\delta x^2\right\} = \pm\,\Sigma m\left\{\frac{1}{r^3}-\frac{3\delta x^2}{r^5}\right\}$$

$$= \pm\,\Sigma m\left\{\frac{1}{r^3}-\frac{\delta x^2+\delta y^2+\delta z^2}{r^5}\right\}.$$

Since $\Sigma mf(r)\delta x^2 = \Sigma mf(r)\delta y^2 = \Sigma mf(r)\delta z^2$

in consequence of the symmetry: and this last expression

$$= \pm\,\Sigma m\left\{\frac{1}{r^3}-\frac{1}{r^3}\right\},$$

which is zero (observing that r does not become ∞ for any particle under the sign Σ, since the displaced particle is not included under the sign Σ).

Hence C is zero if the molecular force be an attractive or repulsive force varying inversely as the square of the distance; and this is evidently true no matter how many different kinds of particles compose the system. Hence it is not likely that the molecular force is an attractive or repulsive force, varying inversely as the square of the distance.

There is therefore good reason for supposing that C is not zero in the equations (D), and we shall accordingly proceed upon that supposition.

§ 17. I now proceed to prove two remarkable and very general theorems respecting transverse and normal vibrations, by the help of which the equations (D) may be reduced to very simple forms. I believe that these theorems, at least the first of them, is capable of very important applications.

§ 18. *To shew that the condition of the vibrations being transversal is*

$$\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0.$$

Let the equation to any surface in which all the particles are in the same phase of vibration (i. e. any wave-surface) be

$$F(x, y, z,) = u,$$

where u is a parameter which does not vary as long as x, y, z belong to the same wave-surface, but is different for different wave-surfaces: for example, if the wave-surface be spherical we may take u to represent the radius, or if it be plane we may take u to represent the perpendicular upon it from the origin. In the former case the above equation would be

$$x^2 + y^2 + z^2 = u^2,$$

and in the latter

$$px + qy + sz = u,$$

where p,. q, s, represent the cosines of the angles which the surface makes with the co-ordinate planes. It is evident that in both these cases u varies only when we pass from one wave-surface to another.

Now supposing that t is constant, the phase of vibration, or what is the same thing, α, β, γ can vary only when u varies, hence α, β, γ must be functions of u and t only: moreover, we may suppose u and t to alter in such a manner that the phase of vibration shall not alter, that is, we may suppose that du and dt are so taken that $d\alpha$, $d\beta$, $d\gamma$ are each zero; hence, supposing du and dt thus taken, we have (remembering that α, β, γ are functions of u and t alone).

$$\frac{d\alpha}{dt}dt + \frac{d\alpha}{du}du = 0 \;\ldots\; (1).$$

$$\frac{\delta\beta}{dt}dt + \frac{d\beta}{du}du = 0 \;\ldots\; (2).$$

$$\frac{d\gamma}{dt}dt + \frac{d\gamma}{du}du = 0 \;\ldots\; (3).$$

Now $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$, are proportional to the cosines of the angles which the normal to the wave-surface makes with the axes of x, y, z, respectively; also $\frac{d\alpha}{dt}, \frac{d\beta}{dt}, \frac{d\gamma}{dt}$, are the velocities of any particle parallel respectively to the axes of x, y, z; if the vibrations be transversal, the sum of these velocities resolved along the normal to the wave-surface must be zero; i. e. we must have

$$\frac{d\alpha}{dt}\frac{du}{dx} + \frac{d\beta}{dt}\frac{du}{dy} + \frac{d\gamma}{dt}\frac{du}{dz} = 0 \ldots$$

hence we evidently obtain (multiplying (1), (2), (3) by $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$, respectively and adding) the following equation,

$$\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0,$$

which is the condition of the vibrations being transversal.

§ 19. *To shew that the conditions of the vibrations being normal, are*

$$\frac{d\alpha}{dy} = \frac{d\beta}{dx}, \quad \frac{d\beta}{dz} = \frac{d\gamma}{dy}, \quad \frac{d\gamma}{dx} = \frac{d\alpha}{dz}.$$

If the vibrations be normal, it is evident that we must have

$$\frac{\text{vel. resolved parallel to axis of } x}{\ldots\ldots \text{do.} \ldots\ldots\ldots\ldots \text{do.} \ldots\ldots\ldots\ y} = \frac{\text{cos angle made by normal and axis of } x}{\ldots\ldots \text{do.} \ldots\ldots\ldots\ldots\ldots\ldots \text{do.} \ldots\ldots\ldots\ldots\ y},$$

$$\text{i. e. } \frac{\frac{d\alpha}{dt}}{\frac{d\beta}{dt}} = \frac{\frac{du}{dx}}{\frac{du}{dy}}.$$

Hence we evidently have from the equations (1) and (2),

$$\frac{d\alpha}{du}\frac{du}{dy} = \frac{d\beta}{du}\frac{du}{dx}, \quad \text{or } \frac{d\alpha}{dy} = \frac{d\beta}{dx},$$

and similarly we may prove that

$$\frac{d\beta}{dz} = \frac{d\gamma}{dy}, \quad \text{and } \frac{d\gamma}{dx} = \frac{d\alpha}{dz}.$$

§ 20. *To adapt the equations* (D) *to the case of transversal vibrations.*

If the vibrations be transversal, we have by Article (18),

$$\frac{d\beta}{dy} + \frac{d\gamma}{dz} = -\frac{d\alpha}{dx}, \text{ and therefore } \frac{d}{dx}\left(\frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) = -\frac{d^2\alpha}{dx^2};$$

hence the first of the equations (D) becomes

$$\frac{1}{m}\frac{d^2\alpha}{dt^2} = A\frac{d^2\alpha}{dx^2} + B\left(\frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2}\right) - (A - B)\frac{d^2\alpha}{dx^2} - \frac{m_{\prime}}{m}C\alpha,$$

or,
$$\frac{1}{m}\frac{d^2\alpha}{dt^2} = B\left(\frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2}\right) - \frac{m_{\prime}}{m}C\alpha$$

and similarly,

$$\frac{1}{m}\frac{d^2\beta}{dt^2} = B\left(\frac{d^2\beta}{dx^2} + \frac{d^2\beta}{dy^2} + \frac{d^2\beta}{dz^2}\right) - \frac{m_{\prime}}{m}C\beta \quad \ldots\ldots (E),$$

$$\frac{1}{m}\frac{d^2\gamma}{dt^2} = B\left(\frac{d^2\gamma}{dx^2} + \frac{d^2\gamma}{dy^2} + \frac{d^2\gamma}{dz^2}\right) - \frac{m_{\prime}}{m}C\gamma$$

which are the equations (D) adapted to the case of transversal vibrations. Since the first of these equations does not contain β or γ, nor the second α or γ, nor the third α or β, it is evident that *each may be integrated separately, and so* α, β, γ *may be found; which is a very important simplification.*

§ 21. *To adapt the equations* (D) *to the case of normal vibrations.*

If the vibrations be normal, we have by Art. (19),

$$\frac{d^2\beta}{dxdy} = \frac{d}{dy}\left(\frac{d\beta}{dx}\right) = \frac{d}{dy}\left(\frac{d\alpha}{dy}\right) = \frac{d^2\alpha}{dy^2}, \text{ and similarly, } \frac{d^2\gamma}{dxdy} = \frac{d^2\alpha}{dz^2};$$

$$\text{and therefore } \frac{d}{dx}\left(\frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) = \frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2}.$$

Hence the equations (D) evidently become

$$\frac{1}{m}\frac{d^2a}{dt^2} = A\frac{d^2a}{dy^2} + B\left(\frac{d^2a}{dx^2} + \frac{d^2a}{dz^2}\right) + (A - B)\left(\frac{d^2a}{dy^2} + \frac{d^2a}{dz^2}\right) - \frac{m_{\prime}}{m}Ca\,;$$

$$\left.\begin{aligned}\text{or, }\ \frac{1}{m}\frac{d^2a}{dt^2} &= A\left(\frac{d^2a}{dx^2} + \frac{d^2a}{dy^2} + \frac{d^2a}{dz^2}\right) - \frac{m_{\prime}}{m}Ca\\ \text{and similarly,}\qquad&\\ \frac{1}{m}\frac{d^2\beta}{dt^2} &= A\left(\frac{d^2\beta}{dx^2} + \frac{d^2\beta}{dy^2} + \frac{d^2\beta}{dz^2}\right) - \frac{m_{\prime}}{m}C\beta\\ \frac{1}{m}\frac{d^2\gamma}{dt^2} &= A\left(\frac{d^2\gamma}{dx^2} + \frac{d^2\gamma}{dy^2} + \frac{d^2\gamma}{dz^2}\right) - \frac{m_{\prime}}{m}C\gamma\end{aligned}\right\}\ldots\ldots(F),$$

which are the equations (D) adapted to the case of normal vibrations. It is remarkable, that these equations should be of exactly the same form as the equations (E), differing only in having A in the place B.

It is evident that the equations of vibratory motion cannot assume the form

$$\frac{1}{m}\frac{d^2a}{dt^2} = A\left(\frac{d^2a}{dx^2} + \frac{d^2\beta}{dy^2} + \frac{d^2\gamma}{dz^2}\right),\ \text{and similar expressions for } \frac{d^2\beta}{dt^2} \text{ and } \frac{d^2\gamma}{dt^2},$$

unless the vibrations be either altogether normal, or altogether transversal.

§ 22. *To adapt the equations* (E) *to the case of plane waves.*

The equation to the wave-surface in this case will be

$$pn + qy + sz = u.$$

When u is the perpendicular from the origin on the surface, and p, q, s the cosines of the angles it makes with the co-ordinate planes, and therefore $p^2 + q^2 + s^2 = 1$.

Now in this case, since a is a function of u and t, we have

$$\frac{da}{dx} = \frac{da}{du}\frac{du}{dx} = p\frac{da}{du},\ \text{and therefore}\ \frac{d^2a}{dx^2} = p^2\frac{d^2a}{du^2};$$

$$\text{and in the same manner, } \frac{d^2a}{dy^2} = q^2\frac{d^2a}{du^2},\ \frac{d^2a}{dz^2} = s^2\frac{d^2a}{du^2};$$

hence the first of the equations (F) becomes

$$\left.\begin{aligned} \frac{1}{m}\frac{d^2\alpha}{dt^2} &= B\frac{d^2\alpha}{du^2} - \frac{m_{\prime}}{m}C\alpha \\ \text{and similarly,}\quad \frac{1}{m}\frac{d^2\beta}{dt^2} &= B\frac{d^2\beta}{du^3} - \frac{m_{\prime}}{m}C\beta \\ \frac{1}{m}\frac{d^2\gamma}{dt^3} &= B\frac{d^2\gamma}{du^2} - \frac{m_{\prime}}{m}C\gamma \end{aligned}\right\}\ldots\ldots(G).$$

In exactly the same manner the equations (F) may be adapted to the case of plane waves.

§ 23. *To adapt the equations* (E) *to the case of spherical waves.*

The equation to the wave-surface in this case will be

$$x^2 + y^2 + z^2 = u^2,$$

u being now the radius of the surface.

In this case, we have

$$\frac{d\alpha}{dx} = \frac{d\alpha}{du}\frac{du}{dx} = \frac{d\alpha}{du}\frac{x}{u}, \qquad \frac{d^2\alpha}{dx^2} = \frac{d^2\alpha}{du^2}\frac{x^2}{u^2} + \frac{d\alpha}{du}\frac{u^2 - x^3}{u^3},$$

$$\text{and similarly,}\quad \frac{d^2\alpha}{dy^2} = \frac{d^2\alpha}{du^2}\frac{y^2}{u^2} + \frac{d\alpha}{du}\frac{u^2 - y^3}{u^3},$$

$$\frac{d^2\alpha}{dz^2} = \frac{d^2\alpha}{du^2}\frac{z^2}{u^2} + \frac{d\alpha}{du}\frac{u^2 - z^2}{u^3};$$

hence we have

$$\frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2} = \frac{d^2\alpha}{du^2} + \frac{2}{u}\frac{d\alpha}{du} = \frac{1}{u}\frac{d^2(u\alpha)}{du^2};$$

hence the first of the equations (F) may evidently be put in the form,

$$\left.\begin{aligned} \frac{1}{m}\frac{d^2(u\alpha)}{dt^2} &= B\frac{d^2(u\alpha)}{du^2} - \frac{m_{\prime}}{m}Cu\alpha \\ \text{and similarly,}\quad \frac{1}{m}\frac{d^2(u\beta)}{dt^2} &= B\frac{d^2(u\beta)}{du^2} - \frac{m_{\prime}}{m}Cu\beta \\ \frac{1}{m}\frac{d^2(u\gamma)}{dt^2} &= B\frac{d^2(u\gamma)}{du^2} - \frac{m_{\prime}}{m}Cu\gamma \end{aligned}\right\}\quad (H)^*.$$

* From these equations, if we obtain $u\alpha$ in the form $f(u, t)$, we have $\alpha = \frac{1}{u}f(u, t)$, $f(u, t)$, being evidently the value of α for plane waves; let a be the maximum value of $f(u, t)$, then

In exactly the same manner the equations (F) may be adapted to the case of spherical waves.

§ 25. From the results in Art. (22) we have for plane waves in vacuum,

$$\frac{1}{m}\frac{d^2 a}{dt^2} = A\frac{d^2 a}{du^2}, \text{ and similar expressions for } \frac{d^2\beta}{dt^2}, \text{ and } \frac{d^2\gamma}{dt^2},$$

when the vibrations are normal,

$$\text{and } \frac{1}{m}\frac{d^2 a}{dt^2} = B\frac{d^2 a}{du^2}, \text{ and similar expressions for } \frac{d^2\beta}{dt^2}, \text{ and } \frac{d^2\gamma}{dt^2}.$$

Hence it follows that transverse and normal vibrations are propagated, in general, with different velocities, namely, $\sqrt{A}$ and $\sqrt{B}$ respectively.

§ 26. If the medium be capable of transmitting transverse vibrations only, we must have $B > 0$, $A = 0$, or < 0, and, of course, the equilibrium stable. Or by § (11), and note § (16),

$$\Sigma\left\{\frac{1}{r^2}\frac{d(Rr^4)}{dr}\right\} > 0 \ldots\ldots(1), \quad \Sigma\left\{r^{\frac{4}{3}}\frac{d(Rr^{\frac{2}{3}})}{dr}\right\} = 0, \text{ or } < 0 \ldots\ldots(2),$$

$$\text{and } \Sigma\left\{\frac{1}{r^2}\frac{d(Rr^2)}{dr}\right\} > 0 \ldots\ldots(3).$$

A number of laws of force might evidently be found satisfying these conditions.

Two laws of force have been a good deal insisted upon, namely, the inverse square, and the inverse fourth power: these conditions shew that neither of these laws can hold (assuming the theory of transversal vibrations); for if $R = \pm\frac{\mu}{r^2}$, $\frac{d(Rr^2)}{dr} = 0$, and therefore (3) is not satisfied, the equilibrium is neutral: and again, if $R = \pm\frac{\mu}{r^4}$, $\frac{d(Rr^4)}{dr} = 0$, and therefore (1) is not satisfied, the velocity of transverse vibrations is zero. Hence neither of these laws of force will answer.

$\frac{a}{u}$ is the maximum value of a for spherical waves; and therefore $\frac{a^2}{u^2}$, the *intensity* of light diverging from a point, which therefore varies as (distance)$^{-2}$.

If the theory of transverse vibrations be true, R cannot $= -\frac{\mu}{r^2}$, for then B is < 0. *Hence, even waiving the objection of neutral equilibrium, a repulsive molecular force varying as* $(dist.)^{-2}$ *is inadmissable.* As to an *attractive* molecular force varying as $(\text{dist.})^{-2}$, it is clearly out of the question, for particles held together by such a force could not possibly vibrate. It appears to me that these considerations are decisive against the Newtonian Law, unless we abandon the theory of transversal vibrations.

§ 27. Having thus arrived at the necessary equations, I now proceed to make use of them for the purpose of explaining the dispersion of light in passing through a prism.

§ 28. *To shew that, in consequence of the action of the material upon the ethereal particles, different colours must be propagated with different velocities in transparent bodies, supposing the particles to vibrate according to the cycloidal law.*

Let us take the case of plane waves of transversal vibrations; the equations to be used in this case are the equations (G).

Suppose the particles to vibrate according to the cycloidal law; and accordingly put for α the well-known form, $a \sin \frac{2\pi}{\lambda}(vt - u)$, and similar values for β and γ, and we find by substitution in the above equations,

$$-\frac{4\pi^2}{\lambda^2}\frac{v^2}{m} = -\frac{4\pi^2}{\lambda^2}B - \frac{m_{,}}{m}C;$$

and therefore $$v^2 = mB + \frac{m_{,}C}{4\pi^2}\lambda^2;$$

which shews that the velocity of propagation in general depends on the length of the wave. In vacuum however $C = 0$, and therefore the velocity of propagation does not depend on the length of the wave. *Hence the direct action of the particles of matter must produce an alteration in the velocity of light depending on the length of the wave, unless we admit the supposition that* C *is zero, which, as I have shewn, is most improbable.* I need not shew that the consequence of the relation thus established between v and λ will be the dispersion of light; but it is very important to inquire whether the dispersion that would be produced by this relation, if true, follows the same law as that which really takes place.

In the above formula as λ increases v increases (supposing C positive), and therefore the index of refraction μ $\left(\text{which varies as } \frac{1}{v}\right)$ diminishes: now as we pass from the violet to the red rays we know that λ increases, hence our formula for v gives an index of refraction diminishing as we pass from violet to red; and so far it agrees with experiment. If C be negative, of course the reverse is the case; it is easy to see that C's being positive or negative depends on the law of force, and that there are a variety of different laws which will make it positive. (See § 26).

§ 29. *To estimate what effect the motion of the material particles has upon the velocity of propagation.*

To do this, we must add to the equations (B) in Article (6) the equations of motion of the material particles, which, if we denote the force of one particle of matter on another by $m_, r_, \psi (r_,)$, will evidently be

$$\left.\begin{aligned} \frac{d^2 \alpha_,}{dt^2} &= \Sigma_, m_, \left\{\psi(r_,)\delta\alpha_, + \frac{1}{r_,}\psi'(r_,)\,\delta x_,(\delta x_,\delta\alpha_, + \delta y_,\delta\beta_, + \delta z_,\delta\gamma_,)\right\} \\ &+ \Sigma m\left\{\phi(r')\Delta\alpha_, + \frac{1}{r'}\phi'(r')\,\Delta x_,(\Delta x_,\Delta\alpha_, + \Delta y_,\Delta\beta_, + \Delta z_,\Delta\gamma_,)\right\} \\ &\text{and similar expressions for } \frac{d^2\beta_,}{dt^2} \text{ and } \frac{d^2\gamma_,}{dt^2}, \end{aligned}\right\} \ldots (B_,).$$

Now it is easy to see that the six equations (B) and ($B_,$) may be satisfied by assuming

$$\alpha = a\cos k(vt - u), \qquad \beta = b\cos k(vt - u), \qquad \gamma = c\cos k(vt - u),$$
$$\alpha_, = a_,\cos k(vt - u_,), \qquad \beta_, = b_,\cos k(vt - u_,), \qquad \gamma = c_,\cos k(vt - u_,),$$

where $u = px + qy + rz$, $u_, = px_, + qy_, + rz_,$, and $a\ b\ c$, $a_,\ b_,\ c_,$, are disposable constants, which we may determine so that the vibrations of the particles shall be wholly transversal: this will appear by substituting as follows these values for $\alpha\ \beta\ \gamma$, $\alpha_,\ \beta_,\ \gamma_,$, and supposing the vibrations transversal.

When the vibrations are transversal and the waves plane, the part of the equation (B) under the sign Σ reduces to the form $mB\dfrac{d^2\alpha}{du^2}$, as appears from Articles (20) and (22); hence, if we put for α the assumed value $a\cos k(vt-u)$, it appears that the part of the equation (B) under the sign Σ reduces to the form $-mBk^2a\cos k(vt-u)$.

To reduce the part under the sign $\Sigma_{,}$ put for a moment

$$\alpha_{,}-\alpha \text{ (or } \Delta\alpha) = \alpha_{,}-\alpha_2+\alpha_2-\alpha, \text{ where } \alpha_2 = a_{,}\cos k(vt-u),$$

$$= \Delta\alpha_2+\alpha_2-\alpha, \text{ and similar expressions for } \Delta\beta\ \Delta\gamma,$$

and it becomes (since $\alpha_2-\alpha$, $\beta_2-\beta$, $\gamma_2-\gamma$ may be brought outside $\Sigma_{,}$)

$$m_{,}C(\alpha_2-\alpha)+\Sigma_{,}m_{,}\left\{\phi(r')\Delta\alpha_2+\frac{1}{r'}\phi'(r')\Delta x(\Delta x\Delta\alpha_2+\Delta y\Delta\beta_2+\Delta z\Delta\gamma_2)\right\},$$

the part of this expression under the sign $\Sigma_{,}$ is exactly similar to that we have just reduced, having $\Sigma_{,}$, $m_{,}$, ϕ, r', Δx, Δy, Δz, $\Delta\alpha_2$, $\Delta\beta_2$, $\Delta\gamma_2$, instead of Σ, m, f, r, δx, δy, δz, $\delta\alpha$, $\delta\beta$, $\delta\gamma$ respectively; hence it must in the same manner reduce to the form

$$-m_{,}B'k^2a_{,}\cos k(vt-u),$$

$$\text{where } B' = \frac{1}{2}\Sigma_{,}\left\{\phi(r')\Delta x^2+\frac{1}{r'}\phi'(r')\Delta x^2\Delta y^2\right\}.$$

Hence, when we have substituted the assumed values of α, β, γ, $\alpha_{,}$, $\beta_{,}$, $\gamma_{,}$, in the equation B, supposing the vibration transversal, we obtain this result [dividing out $\cos k(vt-u)$] viz:

$$-k^2v^2a = -mBk^2a+m_{,}C(a_{,}-a)-m_{,}B'k^2a_{,}\ldots\ldots\ldots(1);$$

and if we substitute in the same manner in the equations for $\dfrac{d^2\beta}{dt^2}$ and $\dfrac{d^2\gamma}{dt^2}$, we obtain precisely the same result.

Treating the equations ($B_{,}$) in a similar manner we obtain a similar result, namely,

$$-k^2v^2a_{,} = -m_{,}B_{,}k^2a_{,}+mC(a-a_{,})-mB'k^2a\ldots\ldots\ldots(2),$$

$$\text{where } B_{,} = \frac{1}{2}\Sigma\left\{\psi(r_{,})\delta x_{,}^2+\frac{1}{r_{,}}\psi'(r_{,})\delta x_{,}^2\delta y_{,}^2\right\}.$$

Now by giving proper values to v^2 and $\frac{a_{,}}{a}$ in (1) and (2), we may satisfy both these equations, and consequently the six equations of motion; hence the assumed values of $\alpha, \beta, \gamma, \alpha_{,}, \beta_{,}, \gamma_{,}$, satisfy the equations of motion provided the vibrations be transversal, and v^2 and $\frac{a_{,}}{a}$ be so assumed as to satisfy (1) and (2).

It is evident from the same reasoning as that employed in Art. 12., that $B'k^2$ is small compared with C, also by Art. 14., $a_{,}$ is small compared with a; hence, for a first approximation, omitting the terms $m_{,}Ca_{,}$, and $m_{,}B'k^2a_{,}$, we obtain from the equation (1),

$$v^2 = B + \frac{m_{,}}{m}\frac{C}{k^2}\left(k = \frac{2\pi}{\lambda}\right),$$

which is the result previously obtained.

For a second approximation we must retain the term $m_{,}Ca_{,}$, but we may omit $m_{,}B'k^2a_{,}$, and $mB'k^2a$, in (1) and (2), and then we have

$$k^2 a = \frac{Cm_{,}}{v^2 - mB}(a - a_{,}),$$

$$k^2 a_{,} = \frac{Cm}{v^2 - m_{,}B_{,}}(a - a_{,}):$$

subtracting the second of these equations from the first, and dividing out $a - a_{,}$, we find

$$k^2 \text{ or } \frac{4\pi^2}{\lambda^2} = \frac{m_{,}C}{v^2 - mB} + \frac{mC}{v^2 - m_{,}B_{,}} \quad \ldots\ldots\ldots\ldots (3),$$

which is the same relation between v and λ as that which I obtained in the Philosophical Magazine, for March 1842, by a different method.

A third approximation may be easily obtained by retaining the last terms of (1) and (2) and eliminating $\frac{a_{,}}{a}$, which will give a still more exact relation between v and λ.

§ 30. *If* C = 0, k² *divides out of* (1) *and* (2), *and therefore the velocity of propagation has no dependance on the length of the wave.*

Hence it appears, *that the mere motion of the particles of matter cannot produce any dispersion of light.*

§ 31. On account of the smallness of m compared with $m_{,}$, it is evident that the second term of the expression for k^2 in equation (3) is small compared with the first. This confirms what was said in Art. 14., that the *motion* of the material particles cannot produce much effect on the motion of the etherial particles.

§ 32. I now proceed to shew that there is another cause capable of producing dispersion, which is in no way dependent on the supposition that white light consists of undulations of different lengths.

§ 33. *To shew that the velocity of propagation is uniform only when the particles of ether vibrate according to the cycloidal law.*

It is evident that if we suppose α to remain invariable, we have

$$\frac{d\alpha}{dt}dt + \frac{d\alpha}{du}du = 0,$$

and the value of $\frac{du}{dt}$ obtained from this equation is the velocity of propagation: let us suppose it constant and denote it by v, then we have

$$\frac{d\alpha}{dt} = -v\frac{d\alpha}{du}, \text{ and } \frac{d^2\alpha}{dt^2} = -v\frac{d^2\alpha}{du\,dt} = -v\frac{d}{du}\left(-v\frac{d\alpha}{du}\right) = v^2\frac{d^2\alpha}{du^2},$$

hence the equations (G) become

$$(v^2 - mB)\frac{d^2\alpha}{du^2} + m_{,}C = 0,$$

and two similar equations for β and γ.

It is clear that if we combine this equation with the equation

$$\frac{d^2\alpha}{dt^2} = v^2\frac{d^2\alpha}{du^2},$$

the most general value which α admits of is

$$\alpha = \cos k(vt - u - \epsilon) + a'\cos k(vt + u - \epsilon'),$$

when a, b, ϵ, ϵ' are arbitrary constants, and

$$k^2 = \frac{m_{,}C}{v^2 - mB},$$

now this value of a shews that the particles vibrate according to the cycloidal law. Hence it follows, that if the velocity of propagation be constant, the particles of ether must vibrate according to the cycloidal law. This conclusion is also true, if we account for dispersion by means of M. Cauchy's hypothesis of finite intervals, neglecting the action of the material upon the etherial particles; for then we shall have an equation of the form

$$\frac{d^2 a}{dt^2} = mB\frac{d^2 a}{du^2} + mD\frac{d^4 a}{du^4}:$$

treating this equation in the same manner as we have just treated the equations

$$\frac{d^2 a}{dt^2} = mB\frac{d^2 a}{du^2} - m_, Ca,$$

we may arrive at the same result as that just obtained.

§ 34. If we consider the manner in which light *may be* produced by combustion, I think it is not very likely that the particles always vibrate according to the cycloidal law. Let us, for example, take the case of oxygen and hydrogen, and suppose the two gases mixed together in the proper proportion to form water. They will remain in a state of stable equilibrium* so long as we keep the temperature below a certain point, but if we raise the temperature above that point, the equilibrium will become unstable, and on the slightest disturbance the particles will rush together to assume new positions of equilibrium. Now they will evidently be unable to assume their positions of equilibrium *immediately*, but will as it were shoot beyond them, and oscillate backwards and forwards for a little time before they come to rest, the consequence of this will be, that an oscillatory motion will be communicated to the etherial particles which surround them, and it may be in this manner that waves of light are produced. Now it is clear, that the material particles will in general communicate their own peculiar kind of vibration, whatever that may be, to the etherial particles, and

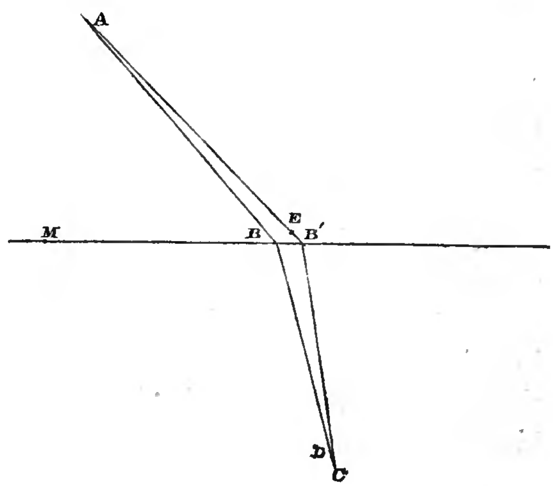
* I speak here of chemical forces, on the supposition that they are the same in kind as common forces, for if this supposition be not true, the results arrived at in the present paper, and all similar results depending upon molecular action, are useless.

the nature of that vibration will depend upon the law of force which acts upon the material particles, and causes them to rush together; that this law should be always such as to produce cycloidal vibrations we have not the least reason to suppose; on the contrary, I conceive it very probable that these vibrations will follow different laws in different kinds of combustion.

Hence we must not assume *a priori* that light is always propagated with a constant velocity in transparent bodies; for I have proved that if the velocity be constant, the particles must necessarily vibrate according to the cycloidal law. What the velocity of propagation is, must depend on the law of vibration; I hope hereafter to investigate the connexion between them. At present I shall only explain generally the effect of a variable velocity of propagation on the refraction of light at a plane surface, and shew that it must cause a dispersion of *homogeneous* light in passing through a prism.

§ 35. *To determine the law of refraction at a plane surface, without assuming that the velocity of propagation is constant.*

Let A be an origin of light, MBB' a plane refracting surface bounding two different media, ABC the course of an elementary disturbance originating at A; i. e. a spherical wave is supposed to spread from A, an element of which takes the course AB, produces a disturbance at B which spreads in a spherical wave from B into the lower medium, an element of which takes the course BC. Let $AB'C$ be the neighbouring course of a similar elementary disturbance which comes from A to C. Take $AE = AB$, $BD = B'C$, then a disturbance takes the same time to travel from A to B, and from A to E, and ultimately



the same time to travel from B to D, and from B' to C, for the paths BD and $B'C$ become ultimately similar in all circumstances when BB' is indefinitely diminished; hence, if τ be the time in which a disturbance travels over the path ABC, τ + time of describing EB' − time of describing DC will be the time in which a disturbance travels over the course $AB'C$; now if v be the velocity of propagation of a wave spreading from A when it arrives at B, v' the velocity of propagation of a wave spreading from B when it arrives at C, it is evident that we have ultimately,

$$\text{time of describing } EB' = \frac{EB'}{v} = \frac{\delta z \sin\phi}{v},$$

$$\text{time of describing } DC = \frac{DC}{v'} = \frac{BC' - B'C}{v'} = \frac{\delta z \sin\phi'}{v'},$$

where ϕ and ϕ' are the angles made by AB and BC respectively with the perpendicular to the refracting surface MBB',

$$\text{and } z = MB, \quad \delta z = BB'.$$

Hence the time in which a disturbance travels over the course $AB'C$ is ultimately

$$\tau + \left(\frac{\sin\phi}{v} - \frac{\sin\phi'}{v'}\right)\delta z,$$

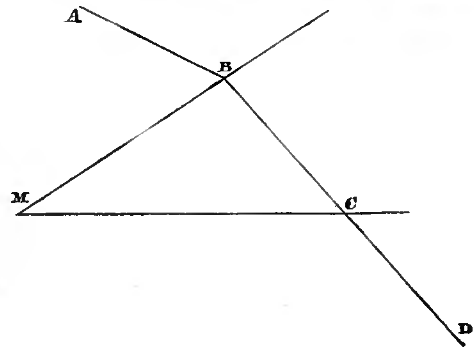
$$\text{and therefore } \delta\tau = \left(\frac{\sin\phi}{v} - \frac{\sin\phi'}{v'}\right)\delta z.$$

If we suppose ABB' to be a small pencil originating at A, we determine its direction after refraction by putting $\delta\tau = 0$ independently of δz (by a well-known principle in Physical Optics): hence the law of refraction of a small pencil is expressed by the formula,

$$\frac{\sin\phi}{v} - \frac{\sin\phi'}{v'} = 0,$$

and this does not suppose that v or v' is constant.

§ 36. In like manner it is easy to shew that if there be two plane refracting surfaces, MB, MC, forming a prism, and if τ denote the time in which a disturbance travels from A to D by the path $ABCD$, ϕ, $\phi_{,}$ the angles which AB and BC make with the perpendicular to MB, and ψ, ψ' those which CD and BC make with the perpendicular to MC, $MB = z$ $MC = z'$; then we have



$$\delta\tau = \left(\frac{\sin\phi}{v} - \frac{\sin\phi'}{v'}\right)\delta z + \left(\frac{\sin\psi'}{v'} - \frac{\sin\psi}{v''}\right)\delta z',$$

where v is the velocity at B, v' at C, and v'' at D.

Hence, to determine the course of a ray, we have, (putting $\delta\tau = 0$ independently of δz and $\delta z'$)

$$\frac{\sin\phi}{v} - \frac{\sin\phi'}{v'} = 0, \quad \frac{\sin\psi'}{v'} - \frac{\sin\psi}{v''} = 0.$$

§ 37. I shall now apply these formulæ *to determine the course of a homogeneous ray passing from vacuum into a prism and emerging into vacuum again.*

In such a case v is constant, and $v'' = v$, and we have

$$\sin\phi = \frac{v}{v'}\sin\phi',$$

$$\sin\psi = \frac{v}{v'}\sin\psi',$$

and the common equation $\phi' + \psi' = i$, i being the angle of the prism.

v' is in general a function of BC and the time t, for it is the velocity of propagation of a spherical wave originating at B when it arrives at C, which I have shewn to be variable except when the vibrations are cycloidal; also $BC = \dfrac{z\sin i}{\cos(i-\phi')}$; hence $\dfrac{v}{v'}$ is a function of ϕ' and t; let us therefore put $\dfrac{v}{v'} = f(\phi' t)$. It is evident that in general f is a recurring function both of ϕ' and t; for $\frac{v}{v'} = -\frac{v\frac{da}{du}}{\frac{da}{dt}}$, see Art. (33); hence, since a is a recurring function of u and t, $\frac{v}{v'}$ must be so also (at least in general), and therefore, since $u = BC = \frac{z \sin i}{\cos(i - \phi')}$, $f(\phi' t)$ must be a recurring function of ϕ' and t: the increments of ϕ' and t which make $f(\phi' t)$ recur being of course extremely small.

The form of the function f depends on the law of vibration, if we knew f we could determine the course of the ray from the two equations

$$\sin \phi = f(\phi,' t) \sin \phi' \ldots\ldots\ldots\ldots (1),$$

$$\frac{\sin \phi}{\sin \psi} = \frac{\sin \phi'}{\sin(i - \phi')} \ldots\ldots\ldots\ldots (2),$$

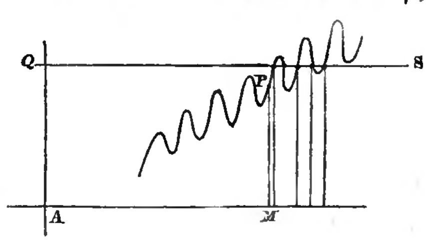
the second equation being obtained from the two first equations at the commencement of this Article, substituting for ψ' its value $i - \phi'$.

Without knowing the form of f we may conclude from its periodical nature that the equation (1) is satisfied by *several different values of* ϕ', *supposing* ϕ *given**, (supposing also for a moment that t is constant,) and then from the equation (2) we may obtain a set of corresponding values of ψ.

Hence it follows, that a single homogeneous ray incident on a prism emerges in several different directions at a given instant.

* This will appear immediately if we construct a curve in which the abscissa $AM = \phi'$, and its ordinate $MP = \phi$.

It is evident that the locus of P will be some such undulating curve as is represented in the figure. If, therefore, we give ϕ one particular value AQ, and draw QS parallel to AM in order to determine the corresponding value or values of ϕ', we shall obtain several different values of ϕ'.



If therefore the emergent light be received on a screen, several spots of light, forming a line of light, will be seen instead of a single spot of light.

I believe that in general these spots will be too close together for the eye to perceive them distinct from each other, therefore nothing but a line of light will be seen. But these spots will in many cases lie closer together in some parts of the line than at others, and, consequently, a variation of the intensity of the light along the line, more or less considerable, will be perceptible.

In what has been said t has been supposed constant: it is easy to see what the effect of the variation of t will be; the values of ϕ' which satisfy the equation (1) will suffer an extremely small periodical variation as t increases; for if we add to t any small increment less than that which causes f to recur, then we may add to ϕ' another small increment less than that which causes f to recur, just sufficient to leave $f(\phi' t)$ unaltered, and at the same time not to make any sensible change in sin ϕ', (for the increment which makes f recur cannot produce any sensible change in ϕ'), and thus the equation (1) will still be satisfied; which shews that as t increases the values of ϕ' which satisfy (1) suffer an extremely small periodical variation, going through all values, when t increases by that increment which makes f recur.

Hence the spots on the screen will perform extremely small and rapid oscillations; this will only spread them into minute lines of imperceptible length. Hence it is evident that the variation of t will produce no sensible alteration in the emergent light.

That the variation of the intensity of light along the line on the screen, taking into account the mixture of different colours, will in many cases be sufficient to produce the appearance of decided interruptions, I hope to shew in a future paper.

What has been said is sufficient, I think, to prove that if the vibrations be not cycloidal, there must be a dispersion of homogeneous light in passing through a prism.

§ 38. *We may find the extent of this dispersion as follows:*

Let μ' be the greatest value of $f(\phi' t)$, and let $\phi_{,}$ and $t_{,}$ be the values of ϕ' and t which satisfy the equations

$$\sin \phi = f(\phi_{,}t_{,}) \sin \phi_{,}, \qquad f(\phi_{,}t_{,}) = \mu_{,},$$

then $\phi_{,}$ is an angle at which light emerges at the time t; also, it is the least angle; for suppose $\phi_{,}$ to be diminished by any quantity, then $\sin \phi_{,}$ is diminished, and $f(\phi_{,}t_{,})$ is not increased, since it is the greatest value of $f(\phi_{,}t)$; therefore $f(\phi_{,}t_{,}) \sin \phi_{,}$ is made less than $\sin \phi$, and consequently, no value of ϕ' less than $\phi_{,}$ will satisfy the equation (1) at the time t. Hence if we put $\phi_{,}$ in the equation (2) and find the corresponding value of ψ ($\psi_{,}$ suppose), $\psi_{,}$ will be the least angle at which light emerges at the time $t_{,}$, and it may be considered the least angle at any other time, since, as we have seen, the variation of t does not produce any perceptible change in the values of ϕ' which satisfy (1).

Hence the equations

$$\sin \phi = \mu_{,} \sin \phi_{,}, \qquad \frac{\sin \phi}{\sin \psi_{,}} = \frac{\sin \phi_{,}}{\sin (i - \phi_{,})},$$

will give us $\psi_{,}$, the least angle at which the light emerges from the prism.

In the same manner it may be proved that if μ_2 be the least value of $f(\phi_{,}t)$, and ψ_2 be obtained from the equations

$$\sin \phi = \mu_2 \sin \phi_2, \qquad \frac{\sin \phi}{\sin \psi_2} = \frac{\sin \phi_2}{\sin (i - \phi_2)},$$

ψ_2 will be the greatest angle at which the light emerges from the prism.

Thus we may obtain $\psi_2 - \psi_1$, which will be the whole dispersion produced by the prism in a homogeneous ray. As $\mu_1 - \mu_2$ is small, we easily obtain the following value of $\phi_2 - \phi_1$ by the common method, viz.

$$\phi_1 - \phi_2 = \frac{\sin i}{\cos \psi \cos \phi_{,}} (\mu_1 - \mu_2).$$

By this formula if we knew the value of $\mu_1 - \mu_2$ we might find the dispersion $\phi_1 - \phi_2$ of a homogeneous ray; and *vice versa*, if we determine $\phi_1 - \phi_2$ by experiment, we shall then know $\mu_1 - \mu_2$, i.e. the difference between the greatest and least values of $\frac{v}{v'}$, and therefore the whole amount of the variation of the velocity of light within the prism.

§ 39. I am not aware whether any experiments have been made which would enable us to determine $\phi_1 - \phi_2$. Any analysis of the spectrum by *second prism is inconclusive* on this point, as I shall endeavour to prove hereafter. I suppose that delicate experiments on the interference of a very small portion of the light composing a very pure spectrum, would enable us to determine whether $\phi_1 - \phi_2$ has any sensible magnitude or not.

The remarkable appearances exhibited when the spectrum is viewed through coloured glasses, seem to indicate pretty clearly that homogeneous light suffers dispersion in passing through a prism.

If experiment shews that $\phi_1 - \phi_2$ is insensible, then we have positive proof that the law of vibration is cycloidal, which is a most important result if true, especially if we bear in mind what has been just proved, namely, that otherwise v must be variable. But if experiment shews that $\phi_1 - \phi_2$ has any magnitude, then we have to take into account the variation of v in all cases of refraction; and the fact, that the law of vibration is not cycloidal.

§ 40. It remains now to prove, that the results thus obtained on the hypothesis of symmetrical arrangement are equally true when the arrangement is unsymmetrical, in consequence of the position of equilibrium of the etherial particles being altered by the action of the material particles; supposing that several etherial surround each material particle.

It is evident that the equation expanded in § (8), now becomes a linear equation with *variable* coefficients, in the form

$$\frac{d^2 a}{dt^2} = P\frac{da}{dx} + Q\frac{da}{dy} \ldots\ldots R\frac{d\beta}{dx} \ldots\ldots S\frac{d^2 a}{dx^2} \ldots\ldots - Ca,$$

where $P, Q \ldots R \ldots S \ldots C$, are functions of x, y, z, is evidently the same for all particles similarly situated with respect to the particles of matter: hence, P, Q, R, S, C recur when we pass from one particle to another similarly situated with respect to the particles of matters.

§ 41. It will be necessary to recur to the original equations (B), in which I shall suppose the particles of matter fixed, and therefore

$$\alpha_{,} = 0, \quad \beta_{,} = 0, \quad \gamma_{,} = 0.$$

Let us put $\alpha = \bar{\alpha} + \epsilon$, $\beta = \bar{\beta} + \eta$, $\gamma = \bar{\gamma} + \zeta$; when, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are displacements, such as constitute a common wave of light in vacuum, and ϵ, η, ζ the quantities to be added to them in order to satisfy the equations (B). Then, denoting the second member of (B) by $F(\alpha\ \beta\ \gamma)$, we have

$$\frac{d^2\bar{\alpha}}{dt^2} + \frac{d^2\epsilon}{dt^2} = F(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}) + F(\epsilon\ \eta\ \zeta).$$

Now since $\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}$ are displacements, such as constitute a common wave of light in vacuum, we may expand $F(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma})$ as in § (8), neglecting all above second differential coefficients. In the part retained the differential coefficients will be multiplied not by constant quantities, but by periodical coefficients, functions of $x\ y\ z$, which recur in the manner just described. Let $F_1(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma})$ be the value of $F(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma})$, when we put for these coefficients their mean values, and let $F_2(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma})$ be the value of $F(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma})$, when we omit the mean part of each coefficient, and retain only its periodical part,

$$\text{then,}\ F(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}) = F_1(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}) + F_2(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}).$$

In $F_1(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma})$ the differential coefficients are multiplied by constant quantities; and in $F_2(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma})$ they are multiplied by periodical functions which go through all their values when we pass from one particle of matter to another similarly situated with respect to the particles of matter, and whose mean values are zero.

Substituting this value of $F(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma})$, the equations ($B$) become

$$\frac{d^2\bar{\alpha}}{dt^2} + \frac{d^2\epsilon}{dt^2} = F_1(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}) + F_2(\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}) + F(\epsilon\ \eta\ \zeta),$$

and similar equations with reference to the axes of y and z.

Now let us assume, as we evidently may, that

$$\left.\begin{array}{l}\dfrac{d^2\bar{\alpha}}{dt^2} = F_1(\bar{\alpha}\,\bar{\beta}\,\bar{\gamma}),\\ \text{and similar expressions for } \dfrac{d^2\bar{\beta}}{dt^2},\ \dfrac{d^2\bar{\gamma}}{dt^2}\end{array}\right\}\cdots\cdots\cdots (M),$$

and then we shall have

$$\left.\begin{array}{l}\dfrac{d^2\epsilon}{dt^2} = F(\epsilon\,\eta\,\zeta) + F_2(\bar{\alpha}\,\bar{\beta}\,\bar{\gamma}),\\ \text{and similar expressions for } \dfrac{d^2\eta}{dt^2},\ \dfrac{d^2\zeta}{dt^2}\end{array}\right\}\cdots\cdots\cdots (N).$$

The equations (*M*) being linear differential equations with constant coefficients, similar to the equations (*D*), in § (15), we may deduce values of $\bar{\alpha}\,\bar{\beta}\,\bar{\gamma}$ from them similar to those of $\alpha\,\beta\,\gamma$ obtained in the previous part of this paper. It is easy to see that $F_1(\bar{\alpha}\,\bar{\beta}\,\bar{\gamma})$ is of exactly the same form as the second members of the equations (*D*), differing only in having different values of *A*, *B*, and *C*.

The equations (*N*) are the same as the equations (*B*), having the term $F_2(\bar{\alpha}\,\bar{\beta}\,\bar{\gamma})$ added to the first, and similar terms to the other two, and having $\epsilon\,\eta\,\zeta$ in place of $\alpha\,\beta\,\gamma$.

I shall now shew that these equations are satisfied by such values of $\epsilon\,\eta\,\zeta$, that $\bar{\alpha}\,\bar{\beta}\,\bar{\gamma}$ are the *mean values* of $\alpha\,\beta\,\gamma$.

To make myself better understood, I shall suppose the material particles to be placed at the corners of cubes, and call the set of etherial particles which lie in the cube formed by any eight contiguous particles of matter, a *cluster* of particles.

Then, in a state of equilibrium, it is evident that the particles which compose any cluster are arranged symmetrically with respect to the middle point of their cube. Moreover, in any two clusters the arrangement of the particles is the same in all respects, so that one cluster is perfectly similar to another.

§ 42. *To shew from the nature of the case, that the equations of motion* (B) *are satisfied by values of $\alpha\ \beta\ \gamma$, whose mean values for all the particles in any cluster are zero, and the same is true when we add to these equations any terms whose mean values for any cluster are zero, and which are the same, or not sensibly different, for similar particles in any two clusters within the sphere of mutual action.*

Conceive the particles of each cluster to be placed a little out of their positions of equilibrium, in such a manner that each cluster is still symmetrically arranged with respect to the center of its cube, and that all the clusters are still perfectly similar to each other; then it is evident, that if the particles be let go, an oscillatory motion will take place (supposing the equilibrium stable), such that each cluster is always symmetrically arranged with respect to the center of its cube, and all the clusters are always perfectly similar to each other.

Hence we may conclude that the equations of motion (B) are satisfied by such values of $\alpha\ \beta\ \gamma$, that the center of gravity of each cluster has no motion; i.e. that the mean values of $\alpha\ \beta\ \gamma$ for all the particles in any cluster are zero.

The same is true if we add to the second members of the three equations (B) any terms being functions of x, y, z, t, whose mean values for all the particles of any cluster are zero, and which are the same for similar particles in any two clusters within the sphere of mutual action.

This is evident from the fact that the terms represent forces which clearly have no effect on the center of gravity of any cluster, and which, being the same for similar particles in any two clusters within the sphere of mutual action, do not derange the similarity of the motion in these two clusters.

It is *only* necessary that these terms should be the same for similar particles in any two clusters *within the sphere of mutual action:* hence if there be a very gradual alteration in these terms as we pass from cluster to cluster, so gradual as not to be perceptible within the sphere of mutual action, it is still true that the equations are satisfied by such values of $\alpha\ \beta\ \gamma$, the mean values of which for any cluster are zero.

§ 43. Now it is evident that in the quantity $F_2(\bar{a}\,\bar{\beta}\,\bar{\gamma})$, $\bar{a}\,\bar{\beta}\,\bar{\gamma}$ vary very slowly indeed as we pass from cluster to cluster, for $\bar{a}\,\bar{\beta}\,\bar{\gamma}$ are displacements which constitute a common wave of light in vacuum, and the length of a wave in vacuum must be very large compared with the intervals between the particles of matter, and therefore the variation of $\bar{a}\,\bar{\beta}\,\bar{\gamma}$ must be extremely small when we pass over a distance equal to the interval between two particles of matter.

Moreover, the coefficients in $F_2(\bar{a}\,\bar{\beta}\,\bar{\gamma})$ are periodical quantities whose mean values for any cluster are zero. Hence, $F_2(\bar{a}\,\bar{\beta}\,\bar{\gamma})$ is a quantity whose mean value for any cluster is zero, and in which the alteration is very gradual as we pass from cluster to cluster, and not perceptible within the sphere of mutual action, (remembering that the sphere of mutual action must be extremely small compared with the length of a wave in vacuum.)

Therefore, if to the first of the equations of motion (*B*) we add the terms $F_2(\bar{a}\,\bar{\beta}\,\bar{\gamma})$, and similar terms to the other two, the equations so formed will be satisfied by values of $a\,\beta\,\gamma$, whose mean values for any cluster are zero. And therefore the equations (*N*) (which only differ from such equations in having $\epsilon\,\eta\,\zeta$ instead of $a\,\beta\,\gamma$,) are satisfied by values of $\epsilon\,\eta\,\zeta$ whose mean values for any cluster are zero.

§ 44. Now if the mean values of $\epsilon\,\eta\,\zeta$ for any cluster be zero, it is evident that $\bar{a}\,\bar{\beta}\,\bar{\gamma}$ are the mean values of $a\,\beta\,\gamma$. Hence it appears that the mean values of $a\,\beta\,\gamma$ may be obtained by expanding the equations (*B*), as in § (8), neglecting all terms involving differential coefficients above the second, and putting for the coefficients of the retained terms their mean values; which process will lead to equations exactly the same in form as those in § (15).

Hence, every thing that has been proved in the previous part of the paper respecting $a\,\beta\,\gamma$, on the supposition of perfect symmetry, is also true of the mean values of $a\,\beta\,\gamma$ for any cluster when the symmetry is disturbed, as it must be, by the action of the material particles.

§ 45. It is evident that $\bar{\alpha}\,\bar{\beta}\,\bar{\gamma}$ represent a wave, or system of waves, regularly transmitted through the ether composing the common refracted light: but $\epsilon\,\eta\,\zeta$ represent a disturbance of quite a different character, propagated with a very slow velocity, and therefore such as makes each cluster (at least, those at or near the bounding surfaces of the transparent body,) an origin of waves spreading into vacuum as if from a point, so that the bounding surfaces will appear to produce light, in the same manner as luminous surfaces.

That the natural colours of bodies, and the absorption of light by coloured media, are the effects of these waves, I hope to shew in a future paper in the following manner, viz. I shall prove that the intensity of the waves represented by $\epsilon\,\eta\,\zeta$ depends on the length of the waves represented by $\bar{\alpha}\,\bar{\beta}\,\bar{\gamma}$; and then, that the intensity of the latter waves depends in general upon the intensity of the former, and thus I shall establish a relation between the intensity of light transmitted through a medium and the length of the wave, such a relation as, I believe, is capable of accounting for the apparently irregular manner in which absorption takes place.

XXIII. *On the Steady Motion of Incompressible Fluids.* By G. G. Stokes, B. A. *Fellow of Pembroke College.*

[Read *April* 25, 1842.]

In this paper I shall consider chiefly the steady motion of fluids in two dimensions. As however in the more general case of motion in three dimensions, as well as in this, the calculation is simplified when $udx + vdy + wdz$ is an exact differential, I shall first consider a class of cases where this is true. I need not explain the notation, except where it may be new, or liable to be mistaken.

To prove that $udx + vdy + wdz$ is an exact differential, in the case of steady motion, when the lines of motion are open curves, and when the fluid in motion has come from an expanse of fluid of indefinite extent, and where, at an indefinite distance, the velocity is indefinitely small, and the pressure indefinitely near to what it would be if there were no motion.

By integrating along a line of motion, it is well known that we get the equation

$$\frac{p}{\rho} = V - \tfrac{1}{2}(u^2 + v^2 + w^2) + C \ldots\ldots\ldots (1),$$

where $dV = Xdx + Ydy + Zdz$, which I suppose an exact differential. Now from the way in which this equation is obtained, it appears that C need only be constant for the same line of motion, and therefore in general will be a function of the parameter of a line of motion. I shall first shew that in the case considered C is absolutely constant, and then that whenever it is, $udx + vdy + wdz$ is an exact differential.

To determine the value of C for any particular line of motion, it is sufficient to know the values of p, and of the whole velocity, at any point along that line. Now if there were no motion we should have

$$\frac{p_1}{\rho} = V + C \;\ldots\ldots\ldots\ldots\ldots\ldots \;(2),$$

p_1 being the pressure in that case. But considering a point in this line at an indefinite distance in the expanse, the value of p at that point will be indefinitely nearly equal to p_1, and the velocity will be indefinitely small. Consequently C is more nearly equal to C_1 than any assignable quantity: therefore C is equal to C_1; and this whatever be the line of motion considered; therefore C is constant.

In ordinary cases of steady motion, when the fluid flows in open curves, it does come from such an expanse of fluid. It is conceivable that there should be only a canal of fluid in this expanse in motion, the rest being at rest, in which case the velocity at an indefinite distance might not be indefinitely small. But experiment shews that this is not the case, but that the fluid flows in from all sides. Consequently at an indefinite distance the velocity is indefinitely small, and it seems evident that in that case the pressure must be indefinitely near to what it would be if there were no motion.

Differentiating therefore (1) with respect to x, we get

$$\frac{1}{\rho}\frac{dp}{dx} = X - u\frac{du}{dx} - v\frac{dv}{dx} - w\frac{dw}{dx};$$

$$\text{but } \frac{1}{\rho}\frac{dp}{dx} = X - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz};$$

$$\text{whence } v\left(\frac{dv}{dx} - \frac{du}{dy}\right) + w\left(\frac{dw}{dx} - \frac{du}{dz}\right) = 0.$$

$$\text{Similarly, } w\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + u\left(\frac{du}{dy} - \frac{dv}{dx}\right) = 0,$$

$$u\left(\frac{du}{dz} - \frac{dw}{dx}\right) + v\left(\frac{dv}{dz} - \frac{dw}{dy}\right) = 0;$$

$$\text{whence } \frac{dv}{dx} = \frac{du}{dy},\quad \frac{dw}{dy} = \frac{dv}{dz},\quad \frac{du}{dz} = \frac{dw}{dx},$$

and therefore $udx + vdy + wdz$ is an exact differential.

When $udx+vdy+wdz$ is an exact differential, equation (1) may be deduced in another way*, from which it appears that C is constant. Consequently, in any case, $udx + vdy + wdz$ is, or is not, an exact differential, according as C is, or is not, constant.

Steady Motion in Two Dimensions.

I shall first consider the more simple case, where $udx + vdy$ is an exact differential. In this case u and v are given by the equations

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \text{ .... } (3),$$

$$\frac{du}{dy} - \frac{dv}{dx} = 0 \text{ .... } (4);$$

and p is given by the equation

$$\frac{p}{\rho} = V - \tfrac{1}{2}(u^2 + v^2) + C.$$

The differential equation to a line of motion is

$$\frac{dy}{dx} = \frac{v}{u}.$$

Now from equation (3) it follows that $udy - vdx$ is always the exact differential of a function of x and y. Putting then

$$dU = udy - vdx,$$

$U = C$ will be the equation to the system of lines of motion, C being the parameter. U may have any value which allows $\frac{dU}{dy}$ and $-\frac{dU}{dx}$ to satisfy the equations which u and v satisfy. The first equation has been already introduced; the second leads to the equation which U is to satisfy; viz.

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = 0 \text{ .... } (5).$$

* See Poisson, *Traité de Mécanique.*

The integral of this equation may be put under different forms. By integrating according to the general method, we get

$$U = F(x + \sqrt{-1}\,y) + f(x - \sqrt{-1}\,y).$$

Now it will be easily seen that U must be wholly real for all values of x and y, at least within certain limits. But $F(a)$ may be put under the form $F_1(a) + \sqrt{-1}\,F_2(a)$, where $F_1(a)$ and $F_2(a)$ are wholly real. Making this substitution in the value of U, we get a result, which, without losing generality, may be put under the form

$$U = F(x + \sqrt{-1}\,y) + F(x - \sqrt{-1}\,y)$$
$$+ \sqrt{-1}\,\{f(x + \sqrt{-1}\,y) - f(x - \sqrt{-1}\,y)\},$$

changing the functions.

If we develope these functions in series ascending according to integral powers of x, by Taylor's Theorem, which can always be done as long as the origin is arbitrary, we get a series which I shall write for shortness,

$$U = 2\cos\left(\frac{d}{dy}\,x\right)F(y) - 2\sin\left(\frac{d}{dy}\,x\right)f(y),$$

the same result as if we had integrated at once by series by Maclaurin's Theorem.

It has been proved that the general integral of (5) may be put under the form

$$U = \Sigma A \epsilon^{ax+\beta y},$$

where $a^2 + \beta^2 = 0$. Consequently a and β must be, one real, the other imaginary, or both partly real and partly imaginary. Putting then $a = a_1 + \sqrt{-1}\,a_2$, $\beta = \beta_1 + \sqrt{-1}\,\beta_2$, introducing the condition that $a^2 + \beta^2 = 0$, and replacing imaginary exponentials by sines and cosines, we find that the most general value of U is of the form

$$U = \Sigma A \epsilon^{n(\cos\gamma x - \sin\gamma y + a)} \,.\, \cos n(\sin\gamma x + \cos\gamma y + b),$$

where A, n, γ, a and b have any real values, the value of U being supposed to be real.

If we take the value of U,

$$U = 2\cos\left(\frac{d}{dy}x\right)F(y) - 2\sin\left(\frac{d}{dy}x\right)f(y),$$

and develope each term, such as ay^a, in $F(y)$ or $f(y)$, in a series, and then sum the series by the formula

$$\cos n\theta + \sqrt{-1}\sin n\theta = \cos^n\theta\left(1 + \frac{n}{1}\sqrt{-1}\tan\theta - \ldots\right),$$

we find that the general value of U takes the form

$$U = \Sigma A r^n \cos(n\theta + B).$$

As long as the origin of y is arbitrary, only integral powers of y will enter into the development of $F(y)$ and $f(y)$, and therefore the above series will contain only integral values of n. For particular positions of the origin however, fractional powers may enter. The equation

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} + \frac{1}{r^2}\frac{d^2U}{d\theta^2} = 0,$$

which (5) becomes when transferred to polar co-ordinates, is satisfied by the above value of U, whatever n be, even if it be imaginary, in which case the value of U takes the form

$$U = \Sigma A r^m \epsilon^{n\theta}\cos(m\theta - \log_\epsilon r^n + B).$$

We may employ equation (5), to determine whether a proposed system of lines can be a system in which fluid can move, the motion being of the kind for which $udx + vdy$ is an exact differential.

Let $f(x,y) = U_1 = C$ be the equation to the system, C being the parameter. Then, if the motion be possible, some value of U which satisfies (5) must be constant for all values of x and y for which U_1 is constant. Consequently this value must be a function of U_1. Let it $= \phi(U_1)$. Then, substituting this value in (5), and performing the differentiations, we get

$$\phi''(U_1)\left\{\left(\frac{dU_1}{dx}\right)^2+\left(\frac{dU_1}{dy}\right)^2\right\}+\phi'(U_1)\left\{\frac{d^2U_1}{dx^2}+\frac{d^2U_1}{dy^2}\right\}=0,$$

$$\text{or,}\quad \frac{\phi''(U_1)}{\phi'(U_1)}+\frac{\dfrac{d^2U_1}{dx^2}+\dfrac{d^2U_1}{dy^2}}{\left(\dfrac{dU_1}{dx}\right)^2+\left(\dfrac{dU_1}{dy}\right)^2}=0 \;\ldots\ldots\ldots\ldots (6).$$

Now, if the motion be possible, the second term of this equation must be a function of U_1; x, y and U_1 being connected by the equation $f(x, y) = U_1$. Consequently, if by means of this latter equation we eliminate x or y from the second term of (6), the other must disappear. If it does not, the motion is impossible; if it does, the integration of equation (6), in which the variables are separated, will give $\phi(U_1)$ under the form

$$\phi(U_1) = AF(U_1) + B,$$

A and B being the arbitrary constants. The values of u and v will immediately be got by differentiation, and then p will be known. Nothing will be left arbitrary but a constant multiplying the values of u and v, and another added to the value of p.

I shall mention a few examples. Let $U = ar^{\frac{1}{2}}\cos\frac{1}{2}\theta$. In this case the lines of motion are similar parabolas about the same focus. The velocity at any point varies inversely as the square root of the distance from the focus.

Again, let $U = axy$. In this case the lines of motion are rectangular hyperbolas about the same asymptotes. Also,

$$u = \frac{dU}{dy} = ax, \text{ and } v = -\frac{dU}{dx} = -ay.$$

In this case therefore the velocity varies as the distance from the centre, and the particles in a section parallel to either of the axes remain in a section parallel to that axis.

I shall now consider the general case, where $udx + vdy$ need not be an exact differential.

In this case p, u and v, are given by the equations

$$\frac{1}{\rho}\frac{dp}{dx} = X - u\frac{du}{dx} - v\frac{du}{dy}, \quad \ldots\ldots\ldots\ldots (7),$$

$$\frac{1}{\rho}\frac{dp}{dy} = Y - u\frac{dv}{dx} - v\frac{dv}{dy}, \quad \ldots\ldots\ldots\ldots (8),$$

$$\frac{du}{dx} + \frac{dv}{dy} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

We still have $\frac{dy}{dx} = \frac{v}{u}$, for the differential equation to a line of motion, where $udy - vdx$ is still an exact differential, on account of equation (9). Eliminating p by differentiation from (7) and (8), and expressing the result in terms of U, we get the equation which U is to satisfy, viz.

$$\frac{dU}{dy}\frac{d}{dx}\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}\right) - \frac{dU}{dx}\frac{d}{dy}\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}\right) = 0,$$

or, for shortness $\left(\frac{dU}{dy}\frac{d}{dx} - \frac{dU}{dx}\frac{d}{dy}\right)\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}\right) = 0. \ldots\ldots\ldots\ldots (10).$

In this case, since $p = \int\left(\frac{dp}{dx}dx + \frac{dp}{dy}dy\right)$, equations (7) and (8) give

$$\frac{p}{\rho} = V - \int\left\{\left(\frac{dU}{dy}\frac{d^2U}{dxdy} - \frac{dU}{dx}\frac{d^2U}{dy^2}\right)dx + \left(\frac{dU}{dx}\frac{d^2U}{dxdy} - \frac{dU}{dy}\frac{d^2U}{dx^2}\right)dy\right\}.$$

Now $\frac{1}{2}d\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2\right\} = \left(\frac{dU}{dx}\frac{d^2U}{dx^2} + \frac{dU}{dy}\frac{d^2U}{dxdy}\right)dx$

$$+ \left(\frac{dU}{dx}\frac{d^2U}{dxdy} + \frac{dU}{dy}\frac{d^2U}{dy^2}\right)dy;$$

whence,

$$\frac{dU}{dy}\frac{d^2U}{dxdy}dx + \frac{dU}{dx}\frac{d^2U}{dxdy}dy = \frac{1}{2}d\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2\right\}$$

$$- \left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}\right)\left(\frac{dU}{dx}dx + \frac{dU}{dy}dy\right);$$

and therefore,

$$\frac{p}{\rho} = V - \tfrac{1}{2}\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2\right\} + \int\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}\right)\left(\frac{dU}{dx}dx + \frac{dU}{dy}dy\right),$$

$$= V - \tfrac{1}{2}(v^2 + u^2) + \int\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}\right)dU.$$

It will be observed that $\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = \chi(U)$, is a first integral of (10). Consequently this latter term, which is the value of C in (1), comes out a function of the parameter of a line of motion as it should.

We may employ equation (10), precisely as before, to enquire whether a proposed system of lines can, under any circumstances, be a system of lines of motion. Let $f(x, y) = U_1 = C$, be the equation to the system; then, putting as before, $U = \phi(U_1)$, we get

$$\phi''(U_1)\left(\frac{dU_1}{dy}\frac{d}{dx} - \frac{dU_1}{dx}\frac{d}{dy}\right)\left\{\left(\frac{dU_1}{dx}\right)^2 + \left(\frac{dU_1}{dy}\right)^2\right\}$$

$$+ \phi'(U_1)\left(\frac{dU_1}{dy}\frac{d}{dx} - \frac{dU_1}{dx}\frac{d}{dy}\right)\left(\frac{d^2U_1}{dx^2} + \frac{d^2U_1}{dy^2}\right) = 0;$$

$$\text{or, } P\phi''(U_1) + Q\phi'(U_1) = 0, \text{ suppose.}$$

Hence, as before, if we express y in terms of x and U_1, from the equation $f(x, y) = U_1$, and substitute that value in $\frac{Q}{P}$, the result must not contain x. If it does, the proposed system of lines cannot be a system of lines of motion; if not, the integration of the above equation will give $\phi(U_1)$, under the form $\phi(U_1) = AF(U_1) + B$, and we can immediately get the values of u, v and p, with the same arbitrary constants as in the previous case.

One case in which the motion is possible is where the lines of motion are a system of similar ellipses or hyperbolas about the same centre, or a system of equal parabolas having the same axis. In the case of the ellipse, the particles in a radius vector at any time remain in a radius vector, and the value of p has the form $\rho V + A + B(x^2 + y^2)$. When however the ellipse becomes a circle, P and Q vanish in the equation $P\phi''(U_1) + Q\phi'(U_1) = 0$. Consequently the form of ϕ may be any whatever. The value of U_1 being $x^2 + y^2$, we have

$$u = 2\phi'(U_1)y, \quad v = -2\phi'(U_1)x;$$

$$\text{whence, } u^2 + v^2 = 4\{\phi'(U_1)\}^2(x^2 + y^2) = 4U_1\{\phi'(U_1)\}^2.$$

Hence, the velocity may be any function of the distance from the centre. It is evident that we may conceive cylindrical shells of fluid, having a common axis, to be revolving about that axis with any velocities whatever, if we do not consider friction, or whether such a mode of motion would be stable. The result is the same if we enquire in what way fluid can move in a system of parallel lines.

In any case where the motion in a certain system of lines is possible, if we suppose two of these lines to be the bases of bounding cylindrical surfaces, and if we suppose the velocity and direction of motion, at each point of a section of the entering, and also of the issuing fluid, to be what that case requires, I have not proved that the fluid *must* move in that system of lines. When the above conditions are given there may still perhaps be different modes of steady motion; and of these some may be stable, and others unstable. There may even be no stable steady mode of motion possible, in which case the fluid would continue perpetually eddying.

In the case of rectangular hyperbolas, the fluid appeared, on making the experiment, to move in hyperbolas when the end at which the fluid entered was broad and the other end narrow, but not when the end by which the fluid entered was narrow. This may, I think, in some measure be accounted for. Suppose fluid to flow out of a vessel where the pressure is p_1 into one where it is p_2, through a small orifice. Then, the motion being steady, we have, along the same line of motion, $\frac{p}{\rho} = C - \frac{1}{2}v^2$, where v is the whole velocity. At a distance from the orifice, in the first vessel, the pressure will be approximately p_1, and the velocity nothing. At a distance in the second vessel, the pressure will be approximately p_2, and therefore the velocity $= \sqrt{\dfrac{2(p_1 - p_2)}{\rho}}$, nearly. The result is the same if forces act on the fluid. Hence the velocity must be approximately constant; and therefore, the fluid which came from the first vessel, instead of spreading out, must keep to a canal of its own of uniform breadth. This is found to agree with experiment. Hence we might expect that in the case of the hyperbolas, if the end at which the fluid entered were narrow, the entering fluid would have a tendency to keep to a canal of its own, instead of spreading out.

In ordinary cases of steady motion, when the lines of motion are open curves, the fluid is supplied from an expanse of fluid, and consequently $udx + vdy + wdz$ is an exact differential. Consequently, cases of open curves for which it is not an exact differential do not ordinarily occur. We may, however, conceive such cases to occur; for we may suppose the velocity and direction of motion, at each point of a section of the entering, and also of the issuing stream, to be such as any case requires, by supposing the fluid sent in and drawn out with the requisite velocity and in the requisite direction through an infinite number of infinitely small tubes.

In the case of closed curves however, in whatever manner the fluid may have been put in motion, it seems probable that, if we neglect the friction against the sides of the vessel, the fluid will have a tendency to settle down into some steady mode of motion. Consequently, taking account of the friction against the sides of the vessel, it seems probable that the motion may in some cases become approximately steady, before the friction has caused it to cease altogether.

Motion symmetrical about an axis, the lines of motion being in planes passing through the axis.

Before considering this case, it may be well to prove a principle which will a little simplify our equations.

The general equations of motion are,

$$\frac{1}{\rho}\frac{dp}{dx} = X - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz}, \qquad (11),$$

$$\frac{1}{\rho}\frac{dp}{dy} = Y - u\frac{dv}{dx} - v\frac{dv}{dy} - w\frac{dv}{dz}, \qquad (12),$$

$$\frac{1}{\rho}\frac{dp}{dz} = Z - u\frac{dw}{dx} - v\frac{dw}{dy} - w\frac{dw}{dz}; \qquad (13).$$

And the equation of continuity is

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \qquad (14).$$

Putting ϖ_1, ϖ_2, ϖ_3, for the last three terms in (11), (12), (13), respectively, we have

$$\frac{p}{\rho} = V - \int(\varpi_1 dx + \varpi_2 dy + \varpi_3 dz).$$

Hence the pressure consists of two parts, the first, ρV, the same as if there were no motion, the second, the part due to the velocity. Now the velocities are given by equation (14), and by the three equations which result on eliminating p from (11), (12), and (13). These latter equations, as well as (14), will be the same as if there were no forces since

$$\frac{dX}{dy} = \frac{dY}{dx}, \quad \frac{dX}{dz} = \frac{dZ}{dx}, \text{ and } \frac{dY}{dz} = \frac{dZ}{dy};$$

and therefore we shall not lose generality by omitting the forces in (11), (12) and (13), since we shall only have to add ρV to the value of p so determined.

When the motion is symmetrical about an axis, and in planes passing through that axis, let z be measured along the axis, and r be the perpendicular distance from the axis, and s be the velocity perpendicular to the axis. Then, transforming the co-ordinates to z and r, and omitting the forces, it will be found that equations (11), (12) and (13) are equivalent to only two separate equations, which are

$$\frac{1}{\rho}\frac{dp}{dr} = -s\frac{ds}{dr} - w\frac{ds}{dz}, \quad\text{.................................}\quad (15),$$

$$\frac{1}{\rho}\frac{dp}{dz} = -s\frac{dw}{dr} - w\frac{dw}{dz}; \quad\text{...............................}\quad (16),$$

and the equation of continuity becomes

$$\frac{ds}{dr} + \frac{s}{r} + \frac{dw}{dz} = 0. \quad\text{......................................}\quad (17).$$

In the case where $udx + vdy + wdz$ is an exact differential, it will be found that the three equations

$$\frac{du}{dy} = \frac{dv}{dx}, \quad \frac{du}{dz} = \frac{dw}{dx}, \quad \frac{dv}{dz} = \frac{dw}{dy},$$

are equivalent to only one equation, which is

$$\frac{ds}{dz} = \frac{dw}{dr} \quad\text{...}\quad (18).$$

In the general case we get, by eliminating p from (15) and (16),

$$\frac{d}{dz}\left(s\frac{ds}{dr} + w\frac{ds}{dz}\right) = \frac{d}{dr}\left(s\frac{dw}{dr} + w\frac{dw}{dz}\right),$$

or

$$\frac{ds}{dr}\frac{ds}{dz} + \frac{ds}{dz}\frac{dw}{dz} + s\frac{d^2 s}{dr\,dz} + w\frac{d^2 s}{dz^2}$$

$$= \frac{dw}{dr}\frac{dw}{dz} + \frac{dw}{dr}\frac{ds}{dr} + w\frac{d^2 w}{dr\,dz} + s\frac{d^2 w}{dr^2} \quad\text{............}\quad (19).$$

The differential equation, between z and r, to a line of motion is

$$\frac{dz}{dr} = \frac{w}{s}.$$

Let μ be a factor which renders $sdz - wdr$ an exact differential, then

$$\frac{d\mu s}{dr} + \frac{d\mu w}{dz} = 0,$$

$$\text{or } \mu\left(\frac{ds}{dr} + \frac{dw}{dz}\right) + s\frac{d\mu}{dr} + w\frac{d\mu}{dz} = 0,$$

or, using (17),

$$s\frac{d\mu}{dr} + w\frac{d\mu}{dz} = \mu\frac{s}{r};$$

whence we easily see that $\mu = r$ is one such factor.

Let then

$$dU = rsdz - rwdr,$$

so that $$s = \frac{1}{r}\frac{dU}{dz}, \quad w = -\frac{1}{r}\frac{dU}{dr}.$$

The equation which U is to satisfy will be got by expressing s and w in terms of U, and substituting in (19) in the general case, or by substituting in (18), in the case where $udx + vdy + wdz$ is an exact differential.

In the latter case the equation which U is to satisfy is

$$\frac{d^2U}{dz^2} + \frac{d^2U}{dr^2} - \frac{1}{r}\frac{dU}{dr} = 0 \text{ .... (20).}$$

In the general case, the equation is what I shall write

$$\left(\frac{dU}{dz}\frac{d}{dr} - \frac{dU}{dr}\frac{d}{dz}\right)\left\{\frac{1}{r^2}\left(\frac{d^2U}{dz^2} + \frac{d^2U}{dr^2} - \frac{1}{r}\frac{dU}{dr}\right)\right\} = 0 \text{...... (21).}$$

The value of p is given by the equation

$$\frac{p}{\rho} = -\int\left\{\left(s\frac{ds}{dr} + w\frac{ds}{dz}\right)dr + \left(s\frac{dw}{dr} + w\frac{dw}{dz}\right)dz\right\}.$$

Now $$\tfrac{1}{2}d(s^2 + w^2) = s\frac{ds}{dr}dr + w\frac{dw}{dz}dz + s\frac{ds}{dz}dz + w\frac{dw}{dr}dr;$$

and therefore

$$\left(s\frac{ds}{dr} + w\frac{ds}{dz}\right)dr + \left(s\frac{dw}{dr} + w\frac{dw}{dz}\right)dz$$

$$= \tfrac{1}{2}d(s^2 + w^2) + \frac{ds}{dz}(wdr - sdz) + \frac{dw}{dr}(sdz - wdr)$$

$$= \tfrac{1}{2}d(s^2 + w^2) + \left(\frac{dw}{dr} - \frac{ds}{dz}\right)\frac{1}{r}dU;$$

$$\text{whence } \frac{p}{\rho} = -\tfrac{1}{2}(s^2 + w^2) + \int\left(\frac{ds}{dz} - \frac{dw}{dr}\right)\frac{1}{r}\,dU$$

$$= -\frac{1}{2r^2}\left\{\left(\frac{dU}{dz}\right)^2 + \left(\frac{dU}{dr}\right)^2\right\} + \int\frac{1}{r^2}\left(\frac{d^2U}{dz^2} + \frac{d^2U}{dr^2} - \frac{1}{r}\frac{dU}{dr}\right)dU \ldots (22).$$

Hence the quantity under the integral sign must be a function of U. And in fact, we can easily shew by trial that

$$\frac{1}{r^2}\left(\frac{d^2U}{dz^2} + \frac{d^2U}{dr^2} - \frac{1}{r}\frac{dU}{dr}\right) = \psi(U)$$

is a first integral of (21). The last term of (22) is the value of the constant in (1).

By expanding U in a series ascending according to integral powers of z, which may be done as long as the origin is arbitrary, it will be found that the integral of (20) may be written under the form

$$U = \cos(\nabla z)\,F(r) + \sin(\nabla z)\,\nabla^{-1}f(r),$$

where $\nabla^2 F(r)$ denotes $\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)F(r)$, and $\nabla^{2n}F(r)$ denotes that the operation $\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}$ is repeated n times on $F(r)$.

We may employ equations (21) or (20) just as before, to determine whether the motion in a proposed system of lines is possible. If $F(r, z) = U_1 = C$ be the equation to the system, we must have, as before, $U = \phi(U_1)$; whence we get, in the general case,

$$\phi''(U_1)\left\{\left(\frac{dU_1}{dz}\frac{d}{dr} - \frac{dU_1}{dr}\frac{d}{dz}\right)\left[\frac{1}{r^2}\left(\frac{dU_1}{dz}\right)^2 + \left(\frac{dU_1}{dr}\right)^2\right]\right\}$$

$$+\,\phi'(U_1)\left\{\left(\frac{dU_1}{dz}\frac{d}{dr} - \frac{dU_1}{dr}\frac{d}{dz}\right)\left[\frac{1}{r^2}\left(\frac{d^2U_1}{dz^2} + \frac{d^2U_1}{dr^2} - \frac{1}{r}\frac{dU_1}{dr}\right)\right]\right\} = 0,$$

and in the more restricted case where $u\,dx + v\,dy + w\,dz$ is an exact differential, we get

$$\phi''(U_1)\left\{\left(\frac{dU_1}{dz}\right)^2 + \left(\frac{dU_1}{dr}\right)^2\right\} + \phi'(U_1)\left(\frac{d^2U_1}{dz^2} + \frac{d^2U_1}{dr^2} - \frac{1}{r}\frac{dU_1}{dr}\right) = 0.$$

As before, the ratio of the coefficients of $\phi''(U_1)$ and $\phi'(U_1)$ must be a function of U_1 alone, when z, r and U_1 are connected by the equation $F(r, z) = U_1$. If the motion be possible, it will in general be determinate, U being of the form $Af(r, z) + B$. If $U = r$ however, the form of ϕ remains arbitrary. In this case the fluid may be conceived to move in cylindrical shells parallel to the axis, the velocity being any function of the distance from the axis.

Particular cases are, where the lines of motion are right lines directed to a point in the axis, and where they are equal parabolas having the axis of z for a common axis. In these cases $udx + vdy + wdz$ is an exact differential.

We may employ equations (20) and (21) to determine whether the hypothesis of parallel sections can be strictly true in any case. In this case, the sections being perpendicular to the axis of z, we must have

$$w = -\frac{1}{r}\frac{dU}{dr} = F(z);$$

$$\frac{dU}{dr} = -rF(z);$$

$$U = -\frac{1}{2}r^2 F(z) + f(z).$$

Substituting this value in (21), we find, by equating to zero coefficients of different powers of r, that the most general case corresponds to

$$U = (a + bz + cz^2)r^2 + ez + f.$$

If $udx + vdy + wdz$ be an exact differential, the most general case corresponds to

$$U = (a + bz)r^2 + c + ez.$$

G. G. STOKES.

Pembroke College, Cambridge,
April, 1842.

XXIV. *On the Truth of the Hydrodynamical Theorem, that if* $udx + vdy + wdz$ *be a Complete Differential with respect to* $x, y, z,$ *at any one instant, it is always so.* *By the Rev.*

J. POWER, M.A., *Fellow and Tutor of Trinity Hall.*

[Read *May* 9, 1842.]

THIS Theorem was first announced by La Grange, who has given a demonstration of it in the *Mécanique Analytique*, Tom. II. p. 307. The late celebrated mathematician, Baron Poisson, has, however, in the last edition of his Mechanics, expressed great doubts of its generality, and has even mentioned that examples have occurred to him in which it is in fault. Those examples, however, he has not given, which is much to be regretted, as the theorem is one of the greatest importance in the theory of fluid motion, and if not generally true, it was highly desirable for the prevention of error, that its want of generality should be placed beyond all doubt, which a single legitimate exception would have been sufficient to effect.

The demonstration of La Grange supposes that the general values $u, v, w,$ the component velocities of any given particle of fluid at the end of the time t, are developable as follows:

$$u = u' + u''t + u'''t^2 + \&c.$$
$$v = v' + v''t + v'''t^2 + \&c.$$
$$w = w' + w''t + w'''t^2 + \&c.$$

and Poisson objects that the demonstration fails when u, v, w are not developable in series of the above form, as may occasionally happen. The objection is a fair and reasonable one; and it is my object in the present communication to shew that even in cases where u, v, w, do not admit of developement in the supposed form, the theorem is nevertheless true.

The general equation of the motion of fluids is

$$dp = \rho\left\{\left(X - \frac{d}{dt}(u)\right) dx + \left(Y - \frac{d}{dt}(v)\right) dy + \left(Z - \frac{d}{dt}(w)\right) dz\right\},$$

which, since

$$\frac{d}{dt}(u) = \frac{du}{dt} + \frac{du}{dx} u + \frac{du}{dy} v + \frac{du}{dz} w,$$

$$\frac{d}{dt}(v) = \frac{dv}{dt} + \frac{dv}{dx} u + \frac{dv}{dy} v + \frac{dv}{dz} w,$$

$$\frac{d}{dt}(w) = \frac{dw}{dt} + \frac{dw}{dx} u + \frac{dw}{dy} v + \frac{dw}{dz} w,$$

becomes

$$-\frac{dp}{\rho} + dV = \left\{\frac{du}{dt} + \frac{du}{dx} u + \frac{du}{dy} v + \frac{du}{dz} w\right\} dx$$

$$+ \left\{\frac{dv}{dt} + \frac{dv}{dx} u + \frac{dv}{dy} v + \frac{dv}{dz} w\right\} dy$$

$$+ \left\{\frac{dw}{dt} + \frac{dw}{dx} u + \frac{dw}{dy} v + \frac{dw}{dz} w\right\} dz,$$

where $dV = Xdx + Ydy + Zdz$.

If we subtract the identity

$$d\left\{\frac{u^2 + v^2 + w^2}{2}\right\} = \left\{u\frac{du}{dx} + v\frac{dv}{dx} + w\frac{dw}{dx}\right\} dx$$

$$+ \left(u\frac{du}{dy} + v\frac{dv}{dy} + w\frac{dw}{dy}\right) dy$$

$$+ \left(u\frac{du}{dz} + v\frac{dv}{dz} + w\frac{dw}{dz}\right) dz,$$

we obtain

$$-\frac{dp}{\rho} + dV - d\left(\frac{u^2+v^2+w^2}{2}\right) = \frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz$$

$$+\left\{v\left(\frac{du}{dy}-\frac{dv}{dx}\right)+w\left(\frac{du}{dz}-\frac{dw}{dx}\right)\right\}dx$$

$$+\left\{u\left(\frac{dv}{dx}-\frac{du}{dy}\right)+w\left(\frac{dv}{dz}-\frac{dw}{dy}\right)\right\}dy$$

$$+\left\{u\left(\frac{dw}{dx}-\frac{du}{dz}\right)+v\left(\frac{dw}{dy}-\frac{dv}{dz}\right)\right\}dz$$

$$=\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz$$

$$+\left(\frac{du}{dy}-\frac{dv}{dx}\right)(vdx-udy)+\left(\frac{du}{dz}-\frac{dw}{dx}\right)(wdx-udz)$$

$$+\left(\frac{dv}{dz}-\frac{dw}{dy}\right)(wdy-vdz).$$

Since ρ is a function of p depending on the nature of the fluid, $\frac{dp}{\rho}$ is a complete differential with respect to x, y, z, as well as the two remaining terms on the left-hand side of the above equation; consequently, putting

$$\alpha = \frac{du}{dy}-\frac{dv}{dx},\quad \beta = \frac{du}{dz}-\frac{dw}{dx},\quad \gamma = \frac{dv}{dz}-\frac{dw}{dy},$$

$$\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz,$$

$$+\alpha(vdx-udy)+\beta(wdx-udz)+\gamma(wdy-vdz),$$

is a complete differential with respect to x, y, z.

If, then, the preceding expression be developed in powers of t, the coefficients of the different powers of t will severally be exact differentials with respect to x, y, z*.

Let u', v', w', denote the value of u, v, w when $t = 0$. Since in all cases of nature the velocities u', v', w' must be finite, it follows that the general values u, v, w will be developable in ascending positive powers of t, integral or otherwise, and the smaller t is taken, the more accurately will these general values be represented by the earlier terms of the series, and it is quite sufficient for our purpose to regard t as indefinitely small.

We may therefore assume

$$u = u' + u''t^{\lambda} + u'''t^{\mu} + \&c.,$$
$$v = v' + v''t^{\lambda} + u'''t^{\mu} + \&c.,$$
$$w = w' + w''t^{\lambda} + w'''t^{\mu} + \&c.,$$

* Let L denote any exact differential with respect to x, y, z, L at the same time containing t.

Suppose that, expanding L in powers of t, we have

$$L = L_1 t^{\alpha} + L_2 t^{\beta} + L_3 t^{\gamma} + \&c.,$$

where α, β, γ, &c., proceed in ascending order from the negative to the positive infinity.

Since the right-hand side is by hypothesis a complete differential with respect to x, y, z, whatever be the value of t, it will continue so when divided by t^{α},

therefore $L_1 + L_2 t^{\beta-\alpha} + L_3 t^{\gamma-\alpha} + \&c.$

is so for all values of t, and consequently when $t = 0$. Therefore L_1 is an exact differential (for since $\beta - \alpha$, $\gamma - \alpha$, are all positive, the remaining terms vanish when $t = 0$).

Hence also $L_2 t^{\beta-\alpha} + L_3 t^{\gamma-\alpha} + \&c.,$

and $L_2 + L_3 t^{\gamma-\beta} + \&c.,$

are exact differentials for all values of t, and therefore when $t = 0$.

Consequently L_2 is so.

In the same way it may be shown that L_3, L_4 are all exact differentials with respect to x, y, z.

If the developement be supposed to contain a term of the form $Mt^m(\log_e t)^n$, it may, if necessary, be demonstrated in a similar manner that M is an exact differential with respect to x, y, z.

the indices λ, μ, &c., being all positive and arranged in ascending order.

It may be observed that the indices are assumed the same in all the three series, which is allowable. For, the series, as exhibited above, may clearly be made to embrace any assignable case by causing to vanish one or more of the coefficients u'', v'', w'', u''', &c. Thus, λ being the lowest index which occurs in any one of the three series, if it occurred only in the first, we should have $v''=0$, $w''=0$; if it occurred in the first and third only, we should have $v''=0$, and so on. The same may be said of μ, the next lowest index which occurs in any one of the three series, and so on for the other indices. But the evanescence of any of these coefficients will not affect the following reasoning. Hence the legitimacy of the assumption is manifest.

If we substitute the developements of u, v, w, in the expressions of α, β, γ, we find

$$\alpha = \alpha' + \alpha'' t^{\lambda} + \alpha''' t^{\mu} + \&c.,$$

$$\beta = \beta' + \beta'' t^{\lambda} + \beta''' t^{\mu} + \&c.,$$

$$\gamma = \gamma' + \gamma'' t^{\lambda} + \gamma''' t^{\mu} + \&c.,$$

$$\text{where } \alpha' = \frac{du'}{dy} - \frac{dv'}{dx}, \qquad \alpha'' = \frac{du''}{dy} - \frac{dv''}{dx}, \ \&c.$$

$$\beta' = \frac{du'}{dz} - \frac{dw'}{dx}, \qquad \beta'' = \frac{du''}{dz} - \frac{dw''}{dx}, \ \&c.$$

$$\gamma' = \frac{dv'}{dz} - \frac{dw'}{dy}, \qquad \gamma'' = \frac{dv''}{dz} - \frac{dw''}{dy}, \ \&c.$$

Thus the expression

$$\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz$$

$$+ \alpha(vdx - udy) + \beta(wdx - udz) + \gamma(wdy - vdz),$$

becomes

$$\lambda t^{\lambda-1}(u''dx + v''dy + w''dz) + \mu t^{\mu-1}(u'''dx + v'''dy + w'''dz) + \&c.$$
$$+ \{\alpha' + \alpha''t^{\lambda} + \alpha'''t^{\mu} + \&c.\}\{(v'dx - u'dy) + (v''dx - u''dy)t^{\lambda} + \&c.\}$$
$$+ \{\beta' + \beta''t^{\lambda} + \beta'''t^{\mu} + \&c.\}\{w'dx - u'dz + (w''dx - u''dz)t^{\lambda} + \&c.\}$$
$$+ \{\gamma' + \gamma''t^{\lambda} + \gamma'''t^{\mu} + \&c.\}\{w'dy - v'dz + (w''dy - v''dz)t^{\lambda} + \&c.\},$$

and, by what has been demonstrated above, the coefficients of all the different powers of t are exact differentials.

Suppose now, that $udx + vdy + wdz$ is an exact differential when $t = 0$, in other words

$$u'dx + v'dy + w'dz$$

is an exact differential, we have then,

$$\frac{du'}{dy} - \frac{dv'}{dx} = 0,$$

$$\frac{du'}{dz} - \frac{dw'}{dx} = 0,$$

$$\frac{dv'}{dz} - \frac{dw'}{dy} = 0.$$

that is, $\alpha' = 0, \quad \beta' = 0, \quad \gamma' = 0.$

Hence it is plain that the term involving the lowest power of t in the above expression is

$$\lambda t^{\lambda-1}(u''dx + v''dy + w''dz),$$

and consequently

$$u''dx + v''dy + w''dz$$

is an exact differential; whence,

$$\alpha'' = 0, \quad \beta'' = 0, \quad \gamma'' = 0.$$

This being the case, it follows that the term involving the next lowest power of t is

$$\mu t^{\mu-1}(u'''dx + v'''dy + w'''dz),$$

and consequently

$$u'''dx + v'''dy + w'''dz$$

is an exact differential, and the demonstration may be carried on as far as we please.

Hence it also follows that

$$(u'dx + v'dy + w'dz) + (u''dx + v''dy + w''dz)t^{\lambda} + \&c.;$$

that is, $(udx + vdy + wdz)$ is an exact differential.

If then, $udx + vdy + wdz$ be an exact differential when $t = 0$, it is so when t is very small; and since the origin of t is arbitrary, by a repetition of the same reasoning we conclude that it is an exact differential at the end of a second very small interval, and so on *ad infinitum*. Hence we conclude that it is so for any finite value of t.

Moreover, since the origin of t is arbitrary, and t may be taken either positively or negatively in the preceding reasoning, it follows that if at any one instant $udx + vdy + wdt$ is an exact differential, it is so at all other instants, past and future.

Hence also it follows that if $udx + vdy + wdz$ is not an exact differential at any one instant, it never will be so during the whole motion, for if it were so at any other instant, it would likewise be so at the former instant.

Having thus, as I conceive, supplied the deficiency of La Grange's demonstration; much as I respect the authority of Poisson, I may be allowed to venture an opinion, that he may have formed a hasty judgement on the cases before him, which he has not thought proper to detail, and which seemed to militate against the generality of this theorem.

I shall conclude with a few remarks in confirmation of the three following consequences of the theorem.

(1) The expression $udx + vdy + wdz$ is in all cases an exact differential when the motion commences from rest.

(2) It is also an exact differential when the initial motion is impressed by pistons impelled with finite velocities, and acting upon the external boundary of the fluid.

(3) It differs from an exact differential by quantities of a higher order than the first, when the motions are extremely small quantities of the 1st order.

In the first case $u'dx + v'dy + w'dz = 0$, which may be regarded as a complete differential with respect to x, y, z, of an arbitrary function of t. The general value $udx + vdy + wdz$ is therefore a complete differential. Or, if we please, we may make $u' = 0$, $v' = 0$, $w' = 0$ in the preceding argument, which does not affect the reasoning, and the conclusion that $udx + vdy + wdz$ is a complete differential will be valid.

In the second case, if the velocities u, v, w be communicated to any point in the interior by pistons acting impulsively on the surface, it follows from D'Alembert's principle, that the impulses at the surface, in conjunction with $-u$, $-v$, $-w$, &c. in the interior, must be subject to the conditions of equilibrium.

Consequently if p be the total reactive pressure sustained at any point in the interior of the fluid during the communication of the velocities u, v, w to that point, we have

$$\frac{dp}{\rho} = -(udx + vdy + wdz),$$

consequently, $udx + vdy + wdz$ is initially a complete differential, and therefore always continues so.

In this reasoning the instantaneous effects of the accelerating forces X, Y, Z, are omitted as vanishing in comparison with the finite impulses, but they may be supposed to act after the initial motion has been communicated, and $udx + vdy + wdz$ will, by the general theorem, continue to be a complete differential.

Thirdly, in the case of very small motions, since

$$\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz$$

differs from $-\frac{dp}{\rho} + dV$, by quantities which involve the products of u, v, w, $\frac{du}{dx}$, &c., it follows that if we limit our approximation to quantities of the first order of these small variables,

$$\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz$$

is a complete differential with respect to x, y, z; and consequently $udx + vdy + wdz$ is a complete differential to the same degree of approximation.

$$\text{For if } \frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz = d.f(x, y, z, t),$$

performing the partial integration with respect to t, we have

$$\begin{aligned} udx + vdy + wdz &= \textstyle\int_t d.f(x, y, z, t) \\ &= d.\textstyle\int_t f(x, y, z, t), \end{aligned}$$

since d and $\int_t$ relate to different variables.

TRINITY HALL,
May 3, 1842.

NOTE.

LA GRANGE, having shown that in very small motions $\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz$ is approximately a complete differential, concludes briefly as follows: "et l'on voit que $\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz$ devant être intégrable relativement à x, y, z, la quantité $udx + vdy + wdz$ devra l'être aussi." This remark, which the author leaves sufficiently obscure, in the conclusion of my paper I endeavoured to put under a clearer point of view.

But since the Paper was printed, I am indebted to my friend Mr STOKES, of Pembroke College, for a remark which convinces me that the conclusion is invalid.

In fact, I had not thought it necessary to exhibit the arbitrary function of x, y, z which ought to be added after the partial integration with respect to t, conceiving it to be implied under the sign $\int_t$, but it it clear that as regards this argument, it ought to be exhibited. Thus the partial integral of

$$\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz = d.f(x, y, z, t),$$

gives $\quad udx + vdy + wdz = \int_t df(x, y, z, t) + u_0dx + v_0dy + w_0dz,$

u_0, v_0, w_0 being functions of x, y, z without t.

And the second side of this equation being put under the form

$$d\int_t (x, y, z, t) + u_0dx + v_0dy + w_0dz,$$

it does not follow that $udx + vdy + wdz$ is a complete differential unless $u_0dx + v_0dy + w_0dz$ be so. But, in general, there does not appear to be sufficient ground for supposing this to be the case.

Or we may reason thus, assuming the series for u, v, w, we have

$$udx + vdy + wdz = u'dx + v'dy + w'dz + t^\lambda.(u''dx + v''dy + w''dz) + \&c.$$

whence $\quad \frac{du}{dt}du + \frac{dv}{dt}dy + \frac{dw}{dt}dz = \lambda.t^{\lambda-1}.(u''dx + v''dy + w''dz) + \&c.;$

and since the left-hand side is a complete differential in very small motions, the right-hand side is so, and consequently $u''dx + v''dy + w''dz$, &c. are so: but we cannot conclude from thence that $udx + vdy + wdz$ is so, unless $u'dx + v'dy + w'dz$ (i.e., the initial value of $udx + vdy + wdz$) be a complete differential.

Hence, except for the above assertion of La Grange, I see no reason to draw any further conclusion for small motions, than has already been drawn for finite motions, namely, that if, for any one value of t, $udx + vdy + wdz$ be a complete differential, it is always so.

J. POWER.

TRINITY HALL, *Nov.* 9, 1842.

ERRATUM. Page 458, line 19 correct thus, '*containing t*'.

ERRATA TO No. XXII.

The Reader is requested to pass over the demonstration given in § 14, as it is erroneous. Another demonstration of the almost self evident proposition enunciated in § 14 is given in § 29, page 423.: in which place, instead of the words:—"Also by Art. 14," (7th line from top) read the following words:—"*Also since the maximum extent of the vibrations of the material particles must be small compared with that of the etherial, for the reasons assigned in Article* 14."

CORRECTIONS TO No. XXIII.

SINCE the publication of the above Paper, a mistake has been pointed out to me by the Rev. J. POWER, in p. 440. The three equations at the bottom of this page are not independent, and therefore the proposition which is the subject of the latter part of this, and of the first paragraph of the following page, is not generally true for motion in three dimensions. In the cases however of motion in two dimensions, and of motion symmetrical about an axis, the three analogous equations are reduced to one, and the proposition is true. None of the succeeding results are affected by this error, excepting that the first paragraph of p. 448 must be restricted to the two cases above mentioned. There also occur the following errata:—

P. 440 l. 3 for C read C_1.

P. 442 l. 13 for x read $\sqrt{-1}\, y$.

P. 445 l. 2 from bottom, insert the terms $-\frac{dU}{dx}\frac{d^2U}{dy^2}dx - \frac{dU}{dy}\frac{d^2U}{dx^2}dy$, on the left-hand side of the equation.

G. G. STOKES.